THE STRONG-BOUNDED TOPOLOGY ON GROUPS OF AUTOMORPHISMS OF A VON NEUMANN ALGEBRA

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0. **Introduction.** Let $\boldsymbol{R}$ be a von Neumann algebra on the Hilbert space $\boldsymbol{H}$, G a topological group, and $a \rightarrow \varphi(a)$ a representation of G as a group of $*$-automorphisms of $\boldsymbol{R}$. Recall that $a \rightarrow \varphi(a)$ is continuous in the strong-bounded topology if and only if

$$\sup_{T \in \boldsymbol{R}, \|T\| \leqq 1} \|[\varphi(a)(T) - T]x\| \rightarrow 0 \qquad (a \rightarrow e)$$

for all $x \in \boldsymbol{H}$. The purpose of this note is to show that for certain von Neumann algebras $\boldsymbol{R}$ and certain groups of $*$-automorphisms $\varphi(a)$, the continuity of $\varphi(a)$ in the strong-bounded topology is a very restrictive condition. For example, if $\boldsymbol{R}$ is abelian and $s \rightarrow \varphi(s)$ is a one-parameter group of $*$-automorphisms of $\boldsymbol{R}$, continuous in the strong-bounded topology, then $\varphi(s)$ is the identity automorphism for all s. If $\boldsymbol{R}$ is either a I_∞ factor or a III_∞ factor (and H is separable), then a strong-bounded continuous one-parameter group of inner automorphisms must be uniformly continuous. Hence, strong-bounded continuous groups of automorphisms are probably not useful in quantum field theory, since the corresponding Hamiltonian operators have bounded spectrum.

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The notation and terminology of this note is that of Dixmier [**1**].

1. **The results.** We make two preliminary remarks before going on to the main theorems.

First, note that if $x \in \boldsymbol{H}$ is cyclic for $\boldsymbol{R}'$ and

$$\sup_{T \in \boldsymbol{R}, \|T\| \leqq 1} \|[\varphi(a)(T) - T]x\| \rightarrow 0 \qquad (a \rightarrow e),$$

then $a \rightarrow \varphi(a)$ is continuous in the strong-bounded topology.

The second remark is contained in the following lemma.

LEMMA 1.1. *Let $\boldsymbol{R}$ be a von Neumann algebra on $\boldsymbol{H}$, G a topological group, and $a \rightarrow \varphi(a)$ a representation of G as a group of $*$-automorphisms*

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of $\boldsymbol{R}$, continuous in the strong-bounded topology. Let $\boldsymbol{S}$ be a von Neumann algebra and ϕ: $\boldsymbol{R}\to\boldsymbol{S}$ a $*$-isomorphism. Then $a\to\phi\circ\varphi(a)\circ\phi^{-1}$ is a representation of G as a group of $*$-automorphisms of $\boldsymbol{S}$, continuous in the strong-bounded topology.

PROOF. By Dixmier [1, Théoréme 3, p. 58], it suffices to consider three special cases: (1) ϕ is an ampliation ϕ: $T\to T\otimes I$, where I is the identity on some Hilbert space $\boldsymbol{K}$; (2) ϕ is an induction, ϕ: $T\to T_E$, where E is a projection in $\boldsymbol{R}'$ with central support equal to I; (3) ϕ is a spatial isomorphism. Cases (2) and (3) are easy and are left to the reader. Case (1) may be reformulated as follows. There exists some index set A such that $\boldsymbol{H}\otimes\boldsymbol{K}=\sum_{\alpha\in A}\oplus\boldsymbol{H}_\alpha$, where for all $\alpha\boldsymbol{H}_\alpha$ is a copy of $\boldsymbol{H}$. ϕ then has the form ϕ: $T\in R\to\sum_{\alpha\in A}\oplus T_\alpha\in\boldsymbol{R}\otimes I$, where $T_\alpha=T$ for all α. Given $x\in\boldsymbol{H}\otimes\boldsymbol{K}$, choose indices $\alpha_1,\cdots,\alpha_n$ such that $x_{\alpha_1}\oplus\cdots\oplus x_{\alpha_n}$ is as close in norm to x as desired. Here x_β is the βth component of x. Then choose δ so that

$$\sup_{|s|\leqq\delta,\,T\in R,\,\|T\|\leqq 1,\,1\leqq i\leqq n}\|[\varphi(s)(T)-T]x_{\alpha_i}\|$$

is as small as desired. Simple inequalities complete the proof. Q.E.D.

THEOREM 1.2. *Suppose $\boldsymbol{R}$ is an abelian von Neumann algebra and $s\to\varphi(s)$ is a one-parameter group of $*$-automorphisms of $\boldsymbol{R}$, continuous in the strong bounded topology. Then $\varphi(s)$ is the identity automorphism for all s.*

Before proving Theorem 1.2, we make a few preliminary remarks and prove a preliminary lemma.

Let φ be a $*$-automorphism of $\boldsymbol{R}$. Recall that φ is freely acting on $\boldsymbol{R}$ if, given a nonzero projection P in $\boldsymbol{R}$, there exists a nonzero projection Q in $\boldsymbol{R}$ such that $Q\leqq P$ and $Q\perp\varphi(Q)$. In general, there exist projections P and Q, fixed under φ, such that φ is the identity on $\boldsymbol{R}_Q$ and is freely acting on $\boldsymbol{R}_P$. For each $\varphi(s)$, let P_s and Q_s be the corresponding P and Q. Note that $P_s=P_{-s}$ and $Q_s=Q_{-s}$. Theorem 1.2 will be proved if we can show $P_s=0$ for all s.

Every abelian von Neumann algebra is $*$-isomorphic to a maximal abelian von Neumann algebra. Hence, by Lemma 1.1 it suffices to consider the case in which $\boldsymbol{R}$ is maximal abelian. In this case the spectral theorem states that $\boldsymbol{R}$ is unitarily equivalent to the multiplication algebra of some measure space (M,μ). From now on we assume that $\boldsymbol{R}$ is such a multiplication algebra.

Next, a lemma needed in the proof of Theorem 1.2.

LEMMA 1.3. *Let* (Ω, ν) *be a measure space,* $f \in L^2(\Omega, \nu)$, *and* φ *a freely acting* $*$*-automorphism of* $L^\infty(\Omega, \nu)$. *Then there exists a projection* $P \in L^\infty(\Omega, \nu)$ *such that* $P \perp \varphi(P)$ *and* $\|Pf\|_2 \geqq \frac{1}{3}\|f\|_2$.

PROOF. By Zorn's Lemma, choose a maximal projection P such that $P \perp \varphi(P)$. P is nonzero since φ is freely acting on $L^\infty(\Omega, \nu)$. Let $Q = P + \varphi(P)$. Claim $\varphi(I-Q) \leqq P$. To show this it suffices to show that $\varphi(I-Q) \leqq Q$. If $R = \varphi(I-Q) \cdot (I-Q) \neq 0$, then $\varphi^{-1}(R) \leqq I-Q \perp Q$ and $R \leqq I - Q \perp Q$. Since φ is freely acting, there exists a nonzero projection $S \leqq \varphi^{-1}(R)$ such that $S \perp \varphi(S)$. But $S \leqq \varphi^{-1}(R) \perp Q$ and $\varphi(S) \leqq R \perp Q$. This contradicts the maximality of P. Hence, $\varphi(I-Q) \leqq P$. Now $I = P + \varphi(P) + (I-Q)$. At least one of $\|Pf\|_2$, $\|\varphi(P)f\|_2$, and $\|(I-Q)f\|_2$ is greater than or equal to $\frac{1}{3}\|f\|_2$. But $I-Q \perp \varphi(I-Q)$, $P \perp \varphi(P)$, and $\varphi(P) \perp \varphi^2(P)$. The result follows. Q.E.D.

Easy examples show that the constant $\frac{1}{3}$ in the above lemma is best possible.

PROOF OF THEOREM 1.2. Let s be a positive real number, Q is a projection in $L^\infty(M, \mu)$ corresponding to a measurable subset of M of finite measure, and χ_Q the characteristic function of Q. Now (P_s, μ) is a measure space and $\varphi(s)$ is a freely acting $*$-automorphism of $L^\infty(P_s, \mu)L^\infty(M, \mu)$. By Lemma 1.3 there exists a projection $R \in L^\infty(P_s, \mu)$ such that

$$\left[\int_{P_s} R\chi_Q d\mu\right]^{\frac{1}{2}} \geqq \frac{1}{3}\left[\int_{P_s} \chi_Q d\mu\right]^{\frac{1}{2}}$$

and $R \perp \varphi(s)(R)$.

$$\begin{aligned}
&\|[\varphi(s)(R) - R]\chi_Q\|_2 \\
&\quad = \|P_s[\varphi(s)(R) - R]\chi_Q\|_2 = \left[\int_{P_s} |\varphi(s)(R) - R|^2 \chi_Q d\mu\right]^{\frac{1}{2}} \\
&\quad = \left[\int_{P_s} [\varphi(s)(R)\chi_Q + R\chi_Q] d\mu\right]^{\frac{1}{2}} \geqq \left[\int_{P_s} R\chi_Q d\mu\right]^{\frac{1}{2}} \\
&\quad \geqq \frac{1}{3}\left[\int_{P_s} \chi_Q d\mu\right]^{\frac{1}{2}} = \frac{1}{3}\mu(P_sQ)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$9 \sup_{|r| \leqq s, T \in \boldsymbol{R}, \|T\| \leqq 1} \|[\varphi(r)(T) - T]\chi_Q\|_2^2 \geqq \mu(P_sQ).$$

If φ is a $*$-automorphism of $\boldsymbol{R}$ and φ leaves a projection R absolutely fixed, then φ^n leaves R absolutely fixed $(n \geqq 1)$. Hence, $Q_{s/n} \leqq Q_s$ and therefore $P_{s/n} \geqq P_s$. From this it follows that

$$\begin{aligned}\mu(P_sQ) &\leqq \mu(P_{s/n}Q)\\ &\leqq 9 \sup_{|r|\leqq s/n,\, T\in \boldsymbol{R},\, \|T\|\leqq 1} \|[\varphi(r)(T) - T]\chi_Q\|_2^2 \to 0 \qquad (n \uparrow +\infty).\end{aligned}$$

Hence, P_s is orthogonal to Q. Since Q's corresponding to measurable sets of finite measure generate $\boldsymbol{R}$, $P_s = 0$ for all s.

As noted above, this proves Theorem 1.2. Q.E.D.

THEOREM 1.4. *Suppose $\boldsymbol{R}$ is either*: (1) *a I_∞ factor*; *or* (2) $\boldsymbol{H}$ *is separable and $\boldsymbol{R}$ is a III_∞ factor. Let $s \to U(s)$ be a strongly continuous one-parameter unitary group in $\boldsymbol{R}$. Let $\varphi(s)(T) = U(s)TU(-s)$ $(T \in \boldsymbol{R})$. Suppose $\varphi(s)$ is continuous in the strong-bounded topology. Then $s \to U(s)$ is uniformly continuous.*

PROOF. Assume $\boldsymbol{R}$ is a I_∞ factor. Let $U(s) = e^{isA}$. $U(s)$ is uniformly continuous if and only if A is bounded. If $U(s)$ is not uniformly continuous, assume that the spectrum of A is unbounded on the positive real axis (otherwise consider the one-parameter unitary group $V(s) = U(-s)$). Let $U(s) = \int_{-\infty}^{+\infty} e^{i\lambda s} dE(\lambda)$. Then there exist integers $n_0 < n_1 < \cdots \uparrow +\infty$ such that $E([n_k, n_k+1)) \neq 0$ $(k \geqq 0)$. Choose a minimal projection $P_0 \leqq E([n_0, n_0+1))$ and $x \in P_0(\boldsymbol{H})$, $\|x\| = 1$. Choose partial isometries U_k $(k \geqq 1)$ such that $U_k^* U_k = P_0$ and $P_k = U_k U_k^* \leqq E([n_k, n_k+1))$. Now since multiplication on the right by $U(s)$ carries the unit ball of $\boldsymbol{R}$ onto itself

$$\sup_{T\in \boldsymbol{R},\, \|T\|\leqq 1} \|[U(s)TU(-s) - T]x\| = \sup_{T\in \boldsymbol{R},\, \|T\|\leqq 1} \|[U(s)T - TU(s)]x\|.$$

$$\begin{aligned}\|[U(s)U_k &- U_kU(s)]x\|\\ &\geqq \|\exp(in_ks)E([n_k, n_k + 1))U_kx - \exp(in_0s)U_kE([n_0, n_0 + 1)x\|\\ &\quad - \|U(s)U_kx - \exp(in_ks)E([n_k, n_k + 1))U_kx\|\\ &\quad - \|U_k \exp(in_0s)E([n_0, n_0 + 1))x - U_kU(s)x\|\\ &\geqq |\exp(in_ks) - \exp(in_0s)| - \sup_{\lambda\in[n_k, n_k+1]} |\exp(i\lambda s) - \exp(in_ks)|\\ &\quad - \sup_{\lambda\in[n_0, n_0+1]} |\exp(i\lambda s) - \exp(in_0s)|\\ &= |\exp(i(n_k - n_0)s) - 1| - 2\sup_{\lambda\in[0,1]} |\exp(i\lambda s) - 1|.\end{aligned}$$

Now $n_k - n_0 \uparrow +\infty$ as $k \uparrow +\infty$. Hence,

$$\limsup_{s\to 0,\, T\in \boldsymbol{R},\, \|T\|\leqq 1} \|[U(s)TU(-s) - T]x\| = 2.$$

CONTRADICTION. Hence, $U(s)$ must be uniformly continuous.

Now assume that R is a III_∞ factor on a separable Hilbert space. The proof of the theorem for this case follows the proof of the I_∞ case almost verbatim. The only twist is the use of the fact that two nonzero projections in a III_∞ factor on a separable Hilbert space are equivalent in the Murray-von Neumann sense. The details are left to the reader. Q.E.D.

We remark that if R is a I_∞ factor on a separable Hilbert space, then the following apparent strengthening of Theorem 1.4 is true. If $\varphi(s)$ is an arbitrary one-parameter group of $*$-automorphisms of R, continuous in the strong-bounded topology, then $s \to \varphi(s)$ is continuous in the norm topology. A sketch of the proof of this follows. $\varphi(s)$ is inner for each s since R is a I_∞ factor. $\varphi(s)$ continuous in the strong-bounded topology implies that $\varphi(s)$ is continuous in the weak-bounded topology. Since H is separable, a result of Kadison ([**2**, Theorem 4.13, p. 195]) now shows that there exists a strongly continuous one-parameter unitary group $s \to U(s)$ in R such that $\varphi(s)(T) = U(s)TU(-s)$. Theorem 1.4 shows $s \to U(s)$ is uniformly continuous. Easy estimates now imply that $s \to \varphi(s)$ is continuous in the norm topology.

It is unknown to the author whether or not an analogue of Theorem 1.4 holds for R a II_∞ factor. The II_1 case is handled by the next theorem.

THEOREM 1.5. *Let R be a II_1 von Neumann algebra and $U(s)$ a strongly continuous one-parameter unitary group in R. Let $\varphi(s)(T) = U(s)TU(-s)$. Then $s \to \varphi(s)$ is continuous in the strong-bounded topology.*

PROOF. There exists some index set B such that $R = \sum_{\beta \in B} \oplus R_\beta$, where each R_β has a faithful finite trace. Note that each $\varphi(s)$ leaves Cent R fixed and hence $\varphi(s) = \sum_{\beta \in B} \oplus \varphi_\beta(s)$, where $\varphi_\beta(s)$ is a one-parameter group of $*$-automorphisms of R_β. An argument like that used in the proof of Lemma 1.1 shows that $\varphi(s)$ is continuous in the strong-bounded topology if each $\varphi_\beta(s)$ is. Hence, we may assume R has a faithful finite trace. By Lemma 1.1, we may assume R has a trace vector x. By the remarks preceeding Lemma 1.1, it suffices to show that

$$\sup_{T \in R, \|T\| \leqq 1} \| [\varphi(s)(T) - T]x \| \to 0 \qquad (s \to 0);$$

i.e.,

$$\sup_{T \in R, \|T\| \leqq 1} \| \varphi(s)(T) - T \|_2 = \| U(s)T - TU(s) \|_2 \to 0 \qquad (s \to 0)$$

(here $\| \cdot \|_2$ denotes the trace norm).

Let P be a "large" spectral projection of

$$U(s) = \int_{-\infty}^{+\infty} e^{i\lambda s} dE(\lambda) \qquad (\text{say } P = E([-n, n]) \text{ for large } n)$$

such that $\|I-P\|_2 < \epsilon$. Let $T \in \boldsymbol{R}$, $\|T\| \leqq 1$. Then

$$\begin{aligned} \|\varphi(s)(T) - T\|_2 &\leqq \|PU(s)PTP - PTPU(s)P\|_2 + 6\epsilon \\ &\leqq 2\|PU(s)P - P\|_\infty + 6\epsilon. \end{aligned}$$

$\|PU(s)P-P\|_\infty \to 0$ $(s \to 0)$ since $PU(s)P$ is a unitary operator with bounded spectrum on $P(\boldsymbol{H})$. Hence, $\|\varphi(s)(T)-T\|_2 \to 0$ $(s \to 0)$.

Q.E.D.

We note that Theorem 1.5 is not true for an arbitrary one-parameter group $\varphi(s)$ of $*$-automorphisms of a II_1 factor. For example, for $n \geqq 1$, let $\boldsymbol{R}_n$ be the algebra of all 2×2 matrices, τ_n the normalized trace on $\boldsymbol{R}_n$, and $\varphi_n(s)$ the $*$-automorphism of $\boldsymbol{R}_n$ given by the unitary $e_0^{ins} 0_{e-ins}$. Further, let $\boldsymbol{S}$ be the C^*-tensor product $\otimes_{n \geqq 1} \boldsymbol{R}_n$, $\tau = \otimes_{n \geqq 1} \tau_n$,

$$\begin{pmatrix} e^{ins} & 0 \\ 0 & e^{-ins} \end{pmatrix}$$

and $\varphi(s) = \otimes_{n \geqq 1} \varphi_n(s)$. If Π_τ is the cyclic representation of $\boldsymbol{S}$ with cyclic vector ξ_τ on the Hilbert space $\boldsymbol{H}_\tau$ corresponding to the state τ, Π_τ is faithful, and $\Pi_\tau(\boldsymbol{S})''$, the strong closure of $\Pi_\tau(\boldsymbol{S})$, is a II_1 factor. Since $\varphi(s)$ leaves τ invariant, there exists a strongly continuous one-parameter unitary group $U(s)$ on $\boldsymbol{H}_\tau$ such that

$$U(s)\Pi(T)U(-s) = \Pi(\varphi(s)(T)) \qquad (T \in \boldsymbol{S}).$$

Easy computations show that

$$\limsup_{s \to 0,\, T \in \Pi_\tau(s),\, \|T\| \leqq 1} \|[U(s)TU(-s) - T]\xi_\tau\| = 2.$$

Bibliography

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