A NOTE ON Z-MAPPINGS AND WZ-MAPPINGS

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1. Introduction. In [2] Isiwata introduced the notion of WZ-mappings which he shows to be an extension of the notion of Z-mappings introduced by Frolik [1]. The purpose of this paper is to show that every Z-mapping defined on the space $X$ is closed if and only if $X$ is normal. Necessary and sufficient conditions are also found so that every WZ-mapping is a Z-mapping.

Throughout this paper, topological spaces are assumed to be completely regular $T_1$-spaces and mappings are continuous functions. If $\phi$ is a mapping taking $X$ into $Y$, then $\Phi$ will denote the Stone extension of $\phi$ taking $\beta X$ into $\beta Y$.

The mapping $\phi: X \to Y$ is a WZ-mapping if $\text{cl}_{\beta Y} \phi^{-1}(y) = \Phi^{-1}(y)$ for each $y$ in the image of $\phi$.

The mapping $\phi: X \to Y$ is a Z-mapping if the images of zero-sets are closed.

The space $X$ has property $Z$ if every closed set $H$ is completely separated from every zero-set disjoint from $H$.

2. Description of a mapping. If $A$ is a closed subset of the space $X$, then let $\phi_A$ denote the natural function taking $X$ onto $Y = X/A$. Let $C$ denote the subcollection of $R^Y$ to which $f$ belongs if and only if $f \circ \phi_A$ is in $C(X)$. Let the topology on $Y$ be the topology induced by $C$, i.e., the smallest topology such that each element of $C$ is continuous. Since $X$ is completely regular, $Y$ is Hausdorff; hence, it follows from [3, Theorem 3.7] that $Y$ is completely regular. Also, by [3, Theorem 3.8], $\phi_A$ is continuous. The referee has pointed out that the topology on $Y$ is the finest completely regular topology such that $\phi_A$ is continuous, $X$ has property $Z$ if and only if $\phi_A$ is a quotient map for each zero-set $A$ in $X$, and $X$ is normal if and only if $\phi_A$ is a quotient map for each closed set $A$ in $X$.

**Lemma.** $\phi_A$ is always a WZ-mapping.

**Proof.** It need only be shown that $\text{cl}_{\beta X} (A) = $ for each closed set $A$ in $X$.

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construction of \( \phi_A, \phi_A(H \cap X) \) and \( \phi_A(K \cap X) \) are disjoint zero-sets in \( Y = X/A \); and so, \( \text{cl}_Y(\phi_A(H \cap X)) \) and \( \text{cl}_Y(\phi_A(K \cap X)) \) are disjoint. It follows that \( \phi_A(A) \) does not belong to \( \text{cl}_Y(\phi_A(H \cap X)) \). Since \( x \in \text{cl}_X(H \cap X) \), \( \Phi_A(x) \in \text{cl}_Y(\phi_A(H \cap X)) \); therefore, \( \Phi_A(x) \neq \phi_A(A) \).

3. Results.

**Theorem 1.** \( X \) has property Z if and only if every WZ-mapping is a Z-mapping.

**Proof.** First, suppose that \( X \) has property Z and that \( \phi \) is a WZ-mapping from \( X \) to \( Y \). Let \( Z \) be a zero-set in \( X \) and \( y \) a point of \( \phi(X) - \phi(Z) \). Then \( \phi^{-1}(y) \) is a closed subset of \( X \). Since \( X \) has property Z, there is a zero-set \( A \) containing \( \phi^{-1}(y) \) that is disjoint from \( Z \). By [3, Theorem 6.5], \( \text{cl}_X(A) \) is disjoint from \( \text{cl}_X(Z) \). Since \( \Phi^{-1}(y) = \text{cl}_X(\phi^{-1}(y)) \), \( \Phi^{-1}(y) \) is a subset of \( \text{cl}_X(A) \); and so, \( y \) is not in \( \Phi(\text{cl}_X(Z)) \), which is a closed set in \( \beta Y \). It follows that \( \phi(Z) \) is closed in \( \phi(X) \).

Now, suppose that \( X \) does not have property Z; that is, suppose that \( H \) is a closed subset of \( X \) and \( K \) is a zero-set disjoint from \( H \) such that \( H \) and \( K \) are not completely separated. By the lemma, \( \phi_H \) is a WZ-mapping. To see that \( \phi_H \) is not a Z-mapping, observe that \( \phi(H) \) is a limit point of \( \phi(K) \).

**Theorem 2.** The space \( X \) is normal if and only if every Z-mapping is closed.

**Proof.** Suppose that \( X \) is not normal.

**Case I.** \( X \) has property Z. Let \( H \) and \( K \) denote disjoint closed sets that are not completely separated. Consider \( \phi_H: X \to Y = X/H \). \( \phi_H \) is not closed since \( \phi_H(H) \) is a limit point of \( \phi_H(K) \). However, by the lemma, \( \phi_H \) is a WZ-mapping and hence a Z-mapping by Theorem 1.

**Case II.** If \( X \) does not have property Z, then there are a closed set \( H \) and a zero-set \( K \) disjoint from \( H \) such that \( H \) and \( K \) are not completely separated. \( \phi_K: X \to X/K \) is not a closed mapping since \( \phi_K(K) \) is a limit point of \( \phi_K(H) \). However, since any two disjoint zero-sets in \( X \) are completely separated, it follows that \( \phi_K \) is a Z-mapping.

Now, in [2] it is shown that a Z-mapping is a WZ-mapping and that if \( X \) is normal, then every WZ-mapping defined on \( X \) is closed. Thus, if \( X \) is normal then every Z-mapping defined on \( X \) is closed.

It should be noted that property Z does not imply normality since any countably compact space has property Z. In particular, if \( \Omega \) denotes the space consisting of the first uncountable segment of the ordinal numbers together with its endpoint \( \omega_1 \), then \( (\Omega \times \Omega) - (\omega_1, \omega_1) \) is a countably compact space that is not normal.
Recall that $X$ is pseudocompact provided that every real-valued mapping defined on $X$ is bounded.

**Theorem 3.** The pseudocompact space $X$ is countably compact if and only if $X$ has property $Z$.

**Proof.** It is obvious that if $X$ is countably compact it has property $Z$. Suppose that $X$ is pseudocompact and has property $Z$ but $X$ is not countably compact. Then there is a set $H = \{P_1, P_2, \cdots\}$, $P_i \neq P_j$ for $i \neq j$, such that $H$ has no limit point. Let $\{U_1, U_2, \cdots\}$ be a collection of mutually exclusive open sets such that $P_i \subseteq U_i$ for each $i$. For each $n$, let $f_n$ be a mapping taking $X$ into $[0, 1]$ such that $f_n(P_n) = 1/n$ and $X - U_n \subseteq f_n^{-1}(0)$. Let $f = \sup \{f_1, f_2, \cdots\}$. $f$ is continuous and $f^{-1}(0)$ does not intersect $H$. By hypothesis, there is a mapping $h: X \to [0, 1]$ such that $H \subseteq h^{-1}(0)$ and $f^{-1}(0) \subseteq h^{-1}(1)$. Let $g = f + h$. Then $1/g$ is a mapping that is unbounded which is a contradiction from which the theorem follows.

From Theorems 1 and 3 the following theorem is obtained.

**Theorem 4.** The pseudocompact space $X$ is countably compact if and only if every $WZ$-mapping defined on $X$ is a $Z$-mapping.

**Bibliography**


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