INEQUALITIES FOR THE EIGENVALUES OF
POWERS OF FUNCTIONS

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In this paper, we give generalizations of a result of A. M. Fink. Specifically, we consider the Sturm-Liouville system

\( (P(x)u')' + \lambda \rho(x)u = 0, \quad x \in (0, l), \)

\( p(0)u'(0) - h_0u(0) = 0, \quad p(l)u'(l) + h_1u(l) = 0, \)

where \( \rho, \rho \) are integrable functions on \([0, l]\) with \( \rho' \) continuous and \( \rho \) positive in a subinterval of \([0, l]\). Furthermore, \( h_0, h_1 \) are nonnegative numbers with \( h_0, h_1 \) approaching \( \infty \) corresponding to the boundary conditions \( u(0) = 0, u(l) = 0 \), respectively. We will be concerned with the dependence of the positive eigenvalues of the system (1), (2) on the function \( \rho \). We denote them accordingly by \( 0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots \).

Fink shows that in the case where (1), (2) reduces to

\( u'' + \lambda \rho(x)u = 0, \quad x \in [0, l], \)

\( u(0) = u(l) = 0, \)

the inequality

\( \lambda_1[\rho^2] \leq (\lambda_1[\rho]/\pi)^2 \)

holds [1]. We prove the following

**Theorem 1.** The positive eigenvalues of (1), (2) satisfy the functional inequality

\( \lambda_{m+n-1}[\rho] \geq (\lambda_m[|\rho_1|^{\alpha}])^{1/\alpha}(\lambda_n[|\rho_2|^{\beta}])^{1/\beta} \)

where \( \rho = \rho_1 \cdot \rho_2 \) and \( 1/\alpha + 1/\beta = 1 \).

When \( \rho_2 \equiv 1 \) on \([0, l]\), \( n = m = 1, k = 1, \alpha = \beta = 2 \), (4) reduces to Fink’s inequality (3). Furthermore, the proof given here generalizes trivially to higher dimensional problems.

**Proof of Theorem.** We make use of the Rayleigh quotient

\( R[\rho, u] = \frac{\int_0^l u''^2 dx + h_0u^2(0) + h_1u^2(l)}{\int_0^l \rho u^2 dx} \)

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and the max-min characterization of the eigenvalues, i.e.,

$$\lambda_{n+1}[p] = \max_{[v_1, \cdots, v_n]} d[v_1, \cdots, v_n],$$

where $[v_1, \cdots, v_n]$ is the subspace spanned by $n$ smooth functions $v_1, \cdots, v_n$ and

$$d[v_1, \cdots, v_n] = \min R[p, u],$$

where the minimum is taken over all smooth functions satisfying the conditions

$$(u, v_k) = \int_0^l p uv_k dx = 0$$

$(k = 1, \cdots, n)$ and $\int_0^l \rho u^2 dx \geq 0$ (see [2, p. 214]). By Hölder's inequality, we have

$$\int_0^l \rho u^2 dx = \int_0^l (\rho_1 u^{2/\alpha})(\rho_2 u^{2/\beta}) dx$$

(8)

$$\leq \left(\int_0^l |\rho_1|^\alpha u^2 dx\right)^{1/\alpha} \left(\int_0^l |\rho_2|^\beta u^2 dx\right)^{1/\beta}$$

with $1/\alpha + 1/\beta = 1$.

Let $U = [u_1, u_2, \cdots, u_{m-1}]$ where $\{u_k\}_{k=1}^{m-1}$ is the set of eigenfunctions corresponding to $\{(\lambda_k[|\rho_1|], \lambda_k[|\rho_2|])\}_{k=1}^{m-1}$ and let $V = [v_1, v_2, \cdots, v_{n-1}]$ where $\{v_k\}_{k=1}^{n-1}$ is the set corresponding to $\{(\lambda_k[|\rho_2|], \lambda_k[|\rho_2|])\}_{k=1}^{n-1}$.

If $W = [u_1, \cdots, u_{m-1}, v_1, \cdots, v_{n-1}]$, then (6) and (7) yield

$$\lambda_{m+n-1}[p] \geq \min_{u \perp W} R[|\rho_1|, u],$$

where the minimum is taken over the set $\{u : u \in \mathcal{C}, u \perp W, \int_0^l \rho u^2 dx \geq 0\}$. By (8) and another application of the max-min principle we have

$$\lambda_{m+n-1}[p] \geq \min_{u \perp W} (R[|\rho_1|, u])^{1/\alpha}(R[|\rho_2|, u])^{1/\beta}$$

$$\geq \left(\min_{u \perp U} R[|\rho_1|, u]\right)^{1/\alpha}\left(\min_{u \perp V} R[|\rho_2|, u]\right)^{1/\beta}$$

$$= (\lambda_{n-1}[|\rho_1|])^{1/\alpha}(\lambda_{n-1}[|\rho_2|])^{1/\beta}.$$ 

Noting that the positive eigenvalues of (1), (2) are also functionals of $p$ and denoting them by $\lambda_1[p] < \lambda_2[p] < \cdots$, we are able to prove the following result.

**Theorem 2.** The positive eigenvalues of (1) with boundary conditions $u(0) = u(l) = 0$ and $\rho(x) > 0$, $x \in [0, l]$ satisfy the inequality
\[ \lambda_{m+n-1}[p] \geq (\lambda_{m-1}[p_1])^{1/\alpha}(\lambda_{n-1}[p_2])^{1/\beta} \]

where \( p = p_1 \cdot p_2, \ p_1 > 0, \ p_2 > 0 \) on \([0, l]\) and \( 1/\alpha + 1/\beta = 1 \).

**Proof.** We note that \( u_r \) is an eigenfunction of (1), (2) corresponding to the eigenvalue \( \lambda_r[p] \) if and only if \( u_r = pu' \) is an eigenfunction of the reciprocal system

\[
\left( \frac{1}{\rho(x)} v' \right)' + \lambda \frac{1}{\rho(x)} v = 0, \quad v'(0) = v'(l) = 0
\]

(9)

corresponding to the eigenvalue \( \lambda_r[p] \). We note, however, that the system (9) has zero for an eigenvalue corresponding to the eigenfunction \( u_0(x) = \text{const} \). The nonzero eigenvalues of (9) satisfy the following maximum-minimum principle:

Let \( D \) denote the space of absolutely continuous functions such that

\[
\int_0^l \frac{1}{\rho} v du_0 dx = \int_0^l \frac{1}{\rho} v dx = 0.
\]

Let \( V_{n-1} \) denote the subspace of \( D \) spanned by the set \( \{v_1, v_2, \ldots, v_{n-1}\} \) taken from \( D \). Denote the Rayleigh quotient corresponding to (9) by

\[
R[p, u] = \frac{\int_0^l \frac{1}{\rho} u'^2 dx}{\int_0^l \frac{1}{\rho} u^2 dx}.
\]

Then \( \lambda_n[p] = \max_{V_{n-1}} \min_{u \in D} R[p, u] \).

The proof now follows as in the proof of Theorem 1.

Finally, we note that the results presented here are related to those in [3] where generalized means and in particular geometric means are discussed. Also, see reference [4].

**References**


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