THE UNIVERSAL COMPACT
SUBUNITHETIC SEMIGROUP

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Compact subunithetic semigroups have been studied in [3] and [4], and related results can be found in [5], [6], and [7].

The structure of compact subunithetic semigroups is completely determined in this paper by exhibiting a universal compact subunithetic semigroup in the continuous homomorphism sense and a universal compact unithetic semigroup in the embedding sense. Generalizations of some of the results of [4] to include nonabelian semigroups are obtained.

If S is a compact [uniquely] divisible semigroup and x ∈ S, then there exists a [unique] minimal compact divisible subsemigroup S(x) of S which contains x. Moreover, each such S(x) is subunithetic [unithetic]. Thus the study of the structure of compact subunithetic semigroups is essential to the study of compact divisible semigroups.

Notation. The following notation will be used throughout this paper:

1. N = set of all positive integers;
2. Q = discrete additive semigroup of positive rationals;
3. I = [0, 1] with usual multiplication and topology;
4. Σ = a-adic solenoid with a = (2, 3, ...) [2, p. 114];
5. Σ* = universal compact solenoidal group [2, 25.19];
6. Φ = universal compact solenoidal semigroup [6, II].

A semigroup S is said to be [uniquely] divisible if for each y ∈ S and each n ∈ N, there exists an [unique] element x ∈ S such that y = x^n. A topological semigroup T is said to be subunithetic if T contains a dense homomorphic image of Q (Note that T is divisible and abelian). A subunithetic semigroup T is said to be unithetic if T is uniquely divisible. If T is a unithetic semigroup and σ: Q → T is a homomorphism such that σ(Q) is dense in T, then the element x = σ(1) is called a unithetic generator of T. (Note that the rational powers of x are dense in T.)

If S is a uniquely divisible topological semigroup and x ∈ S, then the subsemigroup [x] = {x^r: r ∈ Q} (closure in S) is the unithetic subsemigroup of S generated by x. Note that S is unithetic if and only if S = [x] for some x ∈ S.

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The Bohr (or almost periodic) compactification of a topological semigroup $S$ is a pair $(B, \beta)$ such that:

(i) $B$ is a compact semigroup;
(ii) $\beta: S \rightarrow B$ is a continuous homomorphism of $S$ into $B$ such that $\beta(S)$ is dense in $B$; and
(iii) if $f: S \rightarrow T$ is a continuous homomorphism of $S$ into a compact semigroup $T$, then there exists a unique continuous homomorphism $f^*: B \rightarrow T$ such that $f^* \beta = f$.

The existence and uniqueness of the Bohr compactification can be obtained as a consequence of the adjoint functor theorem [8]. Related results can be found in [1]. The semigroup $\Phi$ is the Bohr compactification of the additive semigroup $R$ of nonnegative reals with the usual topology, and $\Sigma^*$ is the Bohr compactification of the group $R \cup (-R)$. Both $\Phi$ and $\Sigma^*$ are unithetic. One of the most essential results in this paper is that $\Sigma^* \times \Phi$ is the Bohr compactification of $Q$.

Lemma 1. Let $h \in \Sigma^*$ and $\{p_n\}$ a sequence of positive prime integers such that $(n+1)! < p_n$ and $p_n + n < p_{n+1}$ for each $n \in N$. Then there exists $g \in \Sigma^*$ such that $\{g^{1/p_n}\}$ converges to $h$.

Proof. We first prove this result for $\Sigma$. Note that $\Sigma$ is the projective limit of copies of the circle group with bonding sequence $(2, 3, 4, \ldots)$. Let $h_0 = (h_1, h_2, \ldots)$ be an element of $\Sigma$. We will construct $g_0 = (g_1, g_2, \ldots)$ in $\Sigma$ inductively such that $\{g_0^{1/p_n}\}$ converges to $h_0$. Let $g_1 = \ldots = g_{p_1-1} = \exp(2\pi i)$. Let $n \in N$, and suppose $g_{p_n-1} = \exp(2\pi i S_n)$ has been defined for some real number $S_n$. Now $h_n = \exp(2\pi i t_n)$ for some real number $t_n$. Then, since $h_0 \in \Sigma$, there exist real numbers $t_1, t_2, \ldots, t_{n-1}$ such that $(j+1)! t_{j+1} = t_1$ and $h_j = \exp(2\pi i t_j)$ for $j = 1, 2, \ldots, h-1$. Let

$$r_n = \frac{t_1 p_n - S_n (p_n - 1)!}{n! p_n}.$$  

Then there exists an integer $k_n$ such that $p_n r_n - 1 \leq k_n \leq p_n r_n + 1$. Since $p_n$ is prime, $((p_n - 1)!) + 1)/p_n$ is an integer, and hence $m_n = k_n ((p_n - 1)! + 1)/p_n$ is an integer. Thus

$$t_1 - n!/p_n \leq (S_n + h! k_n) (p_n - 1)!/p_n - h! m_n \leq t_1 + n!/p_n.$$

It now follows that

$$t_{j+1} - \frac{n!}{p_n (j + 1)!} \leq \frac{(S_n + n! k_n)(p_n - 1)!}{p_n (j + 1)!} - \frac{n! m_n}{(j + 1)!} \leq t_{j+1} + \frac{n!}{p_n (j + 1)!}.$$  

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and \( n! \frac{m_n}{(j+1)!} \) is an integer for \( j = 0, 1, 2, \ldots, n-1 \). Define

\[
g_{p_n+j} = \exp\left(\frac{2\pi i (S_n + n!k_n)(p_n - 1)!}{(p_n + j)!}\right)
\]

for \( j = 0, 1, \ldots, n-1 \), and define \( g_j \) such that \( g_j = g_{j-1} \) for \( j = p_n + n, \ldots, p_{n+1} - 1 \). Thus \( g_0 \)

is defined inductively such that \( g_0 \in \Sigma \). Now

\[
g_{0/p_n} = (g_{p_n}^{p_n}, \ldots, g_{p_n+j}^{(p_n+j)!/p_n})
\]

and

\[
g_{p_n+j}^{(p_n+j)!/p_n} = \exp\left(\frac{2\pi i (S_n + n!k_n)(p_n - 1)!}{p_n(j+1)!}\right)
\]

for \( j = 0, 1, \ldots, n-1 \).

Since \( \{n!/p_n\} \) converges to 0, it follows from the last inequality above that \( \{g_{0/p_n}\} \) converges to \( h_0 \). This proves the result for \( \Sigma \).

Let \( \Sigma_\alpha = \Sigma \) for \( \alpha \in I \). Then \( \Sigma^* = \prod \{\Sigma_\alpha : \alpha \in I\} \). Let \( \pi_\alpha : \Sigma^* \to \Sigma_\alpha \) be the projection map, and \( h_\alpha = \pi_\alpha(h) \) for each \( \alpha \in I \). Then there exists \( g_\alpha \in \Sigma_\alpha \) such that \( \{g_{0/p_n}\} \) converges to \( h_\alpha \) for each \( \alpha \in I \). Let \( g \in \Sigma^* \) such that \( \pi_\alpha(g) = g_\alpha \) for each \( \alpha \in I \). Then \( \{g_{0/p_n}\} \) converges to \( h \), and the proof of the lemma is complete.

**Theorem 2.** The Bohr compactification of \( Q \) is \( \Sigma^* \times \Phi \).

**Proof.** Let \( x \) be a unithetic generator of \( \Phi \) and 1 the identity of \( \Phi \). Let \( h \) be a unithetic generator of \( \Sigma^* \) and \( \{p_n\} \) a sequence of positive prime integers such that \( (n+1)! < p_n \) and \( p_n + h < p_{n+1} \) for each \( n \in N \). Then, by Lemma 1, there exists \( g \in \Sigma^* \) such that \( \{g_{0/p_n}\} \) converges to \( h \). Thus \( \{(g, x)_{0/p_n}\} \) converges to \( (h, 1) \). Since \( h \) is a unithetic generator of \( \Sigma^* \), it follows that \( \Sigma^* \times \{1\} \subseteq [(g, x)] \). Let \( (a, b) \in \Sigma^* \times \Phi \). Then there exists \( u \in \Sigma^* \) such that \( (u, b) \in [(g, x)] \). Let \( u^{-1} \) denote the inverse of \( u \) in \( \Sigma^* \). Then \( (u^{-1}a, 1) \in [(g, x)] \) and hence \( (a, b) = (u, b) \cdot (u^{-1}a, 1) \) is in \( [(g, x)] \). It follows that \( \Sigma^* \times \Phi \) is unithetic, and that \( (g, x) \) is a unithetic generator of \( \Sigma^* \times \Phi \). Define \( \beta : Q \to \Sigma^* \times \Phi \) by \( \beta(r) = (g, x)^r \) for each \( r \in Q \). Then \( \beta \) is a homomorphism such that \( \beta(Q) \) is dense in the compact semigroup \( \Sigma^* \times \Phi \). Let \( S \) be a compact subunithetic semigroup, \( e \) the identity of \( S \), and \( f : Q \to S \) a homomorphism such that \( f(Q) \) is dense in \( S \). Then \( S \) contains a compact solenoidal subsemigroup \( T \) such that \( S = H(e)T \). Define \( \phi : H(e) \times T \to S \) by \( \phi(s, t) = st; s \in H(e), t \in T \). Then \( \phi \) is a continuous onto homomorphism since \( S \) is abelian. Let \( s_0 \in H(e) \) and \( t_0 \in T \) such that \( \phi(s_0, t_0) = f(1) \). Then there exist continuous onto homomorphisms \( \alpha : \Sigma^* \to H(e) \) and \( \lambda : \Phi \to T \) such that \( \alpha(g) = s_0 \) and \( \lambda(x) = t_0 \). Define \( f^* : \beta(Q) \to S \) by \( f^*(\beta(r)) = \phi(\alpha(g^r), \lambda(x^r)) \); \( r \in Q \). Then, since \( \beta(Q) \)
is dense in $\Sigma^* \times \Phi$, $f^*_x$ has a unique extension to $f^*: \Sigma^* \times \Phi \to S$, and $f^* \beta = f$. It follows that $(\Sigma^* \times \Phi, \beta)$ is the Bohr compactification of $Q$.

**Corollary 3.** A compact semigroup is subunithetic if and only if it is a continuous homomorphic image of $\Sigma^* \times \Phi$.

**Corollary 4.** Let $S$ and $T$ be compact semigroups. Then $S \times T$ is sub-unithetic if and only if $S$ and $T$ are sub-unithetic and either $S$ or $T$ is a group.

**Corollary 5.** The semigroup $((\Sigma^* \times I)/(\Sigma^* \times \{0\})) \times \Sigma^*$ is unithetic.

**Theorem 6.** Let $S$ be a compact semigroup. Then these are equivalent:

(i) $S$ is uniquely divisible;

(ii) each component of $S$ is a uniquely divisible subsemigroup of $S$;

(iii) if $x \in S$, then there exists a unique subsemigroup $S(x)$ of $S$, which is minimal with respect to being a compact divisible subsemigroup of $S$ containing $x$; and each such $S(x)$ is unithetic subunithetic.

**Proof.** The proof follows from the fact that $\Sigma^* \times \Phi$ is connected, and, if $S$ is divisible, and $x \in S$, then there exists a continuous homomorphism $\lambda: \Sigma^* \times \Phi \to S$ such that $\lambda(y) = x$, for some unithetic generator $y$ of $\Sigma^* \times \Phi$.

**Notation.** If $S$ is a semigroup, then $E(S)$ denotes the set of idempotent elements of $S$.

**Corollary 7.** Let $S$ be a compact totally disconnected semigroup. Then $S$ is divisible if and only if $S = E(S)$.

**Corollary 8.** A finite semigroup $S$ is divisible if and only if $S = E(S)$.

**Corollary 9.** Let $S$ be a compact semigroup. Then each element of $S \setminus E(S)$ lies on a unique usual thread in $S$ if and only if $S$ is uniquely divisible and has degenerate subgroups.

**Theorem 10.** Let $S$ be a compact unithetic semigroup. Then $S$ is topologically isomorphic to a subsemigroup of $((\Sigma^* \times I)/(\Sigma^* \times \{0\})) \times \Sigma^*$.

**Proof.** Let 1 denote the identity of $S$, $K$ the minimal ideal of $S$, and $h: H(1) \to \Sigma^*$ and $k: K \to \Sigma^*$ injections. (See [3, Theorem 2.3 and Theorem 3.1].) Let $\gamma: S/K \to (H(1) \times I)/(H(1) \times \{0\})$ be a topological isomorphism. (See [3, Theorem 3.4].) Let $j: I \to I$ be the identity map, $\alpha: H(1) \times I \to (H(1) \times I)/(H(1) \times \{0\})$ the natural maps, and $\lambda: \Sigma^* \times I \to (\Sigma^* \times I)/(\Sigma^* \times \{0\})$ the natural map. Then there exists an induced injection $\phi$ such that the diagram:
commutes. Let \( e \) denote the identity of \( K \) and define \( \sigma: S \to (S/K) \times K \) by \( \sigma(x) = (\psi(x), ex); x \in S, \) where \( \psi: S \to S/K \) is the natural map. Then \( \sigma \) is an injection. Let \( \rho: \Sigma^* \to \Sigma^* \) be the identity map. Then

\[
S \xrightarrow{\sigma} (S/K) \times K \xrightarrow{\gamma \times k} \frac{(H(1) \times I)}{(H(1) \times \{0\})} \times \frac{(\Sigma^* \times I)}{(\Sigma^* \times \{0\})} \times \Sigma^*
\]
defines the desired injection, and the proof of the theorem is complete.

References