OSCILLATION OF NONLINEAR SECOND-ORDER EQUATIONS

L. E. BOBISUD

Let $\mathfrak{F}$ denote the set of all solutions of

$$y'' + a(t)f(y) = 0 \tag{1}$$

which exist on some positive half-line $[T_y, \infty)$, where T_y depends on the particular solution y. A solution of (1) is said to be oscillatory if, given $t_0 > 0$, there exists $t > t_0$ such that $y(t) = 0$; the equation itself is said to be oscillatory if each $y \in \mathfrak{F}$ is oscillatory.

With $f(y) \equiv y^{2n+1}$ $(n \geqq 1)$ and $a > 0$, Atkinson [1] proved that all solutions of (1) are oscillatory if and only if

$$\int_0^\infty ta(t)dt = +\infty. \tag{2}$$

For the more general equation (1) with $yf(y) > 0$ for $y \neq 0$ and f' continuous and nonnegative, Waltman [6] proved that all solutions are oscillatory if and only if (2) holds, provided $a > 0$ and

$$\liminf_{y \to \infty} \frac{|f(y)|}{|y|^p} \neq 0$$

for some $p > 1$. Wong [7] has shown that this result is valid without the restriction $f \in C^1$, $f' \geqq 0$.

The aim here is to prove oscillation theorems for (1) without the restriction $a > 0$. Bhatia [2] has established such results under the condition $\int_0^\infty a(s)ds = +\infty$; this condition is here relaxed to (2) for a suitable class of equations of the form (1). Kartsatos [3] has proved a result somewhat akin to Theorem 1 below for the more general equation

$$y'' + a(t)g(y, y') = 0,$$

$a(t)$ not assumed positive, under conditions much different from ours. Kiguradze [4] has shown that continuable solutions of the equation

$$u'' + a(t)|u|^n \operatorname{sgn} u = 0$$

are oscillatory if $n > 1$ and $\int^\infty \phi(t)a(t)dt = \infty$ for a continuous, positive, concave function $\phi(t)$.

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THEOREM 1. *Let f be continuous and continuously differentiable on $(-\infty, 0)\cup(0, \infty)$ with $uf(u)>0, f'\geqq 0$ there; let f also satisfy*

$$\int_1^\infty \frac{du}{f(u)} < \infty, \qquad \int_{-1}^{-\infty} \frac{du}{f(u)} < \infty.$$

Assume that $a(t)$ is locally integrable and that, for each sufficiently large $x>0$,

$$0 < \lim_{t\to\infty} \int_x^t a(s)ds \leqq +\infty.$$

Finally, assume

$$\int_x^\infty sa(s)ds = +\infty.$$

Then every $y\in\mathfrak{F}$ is either oscillatory or tends monotonically to zero as $t\to\infty$.

PROOF. Let $y\in\mathfrak{F}$ be nonoscillatory, so for some $\beta>0$, $y\neq 0$ on $[\beta, \infty)$. Then we can divide (1) by $f(y(s))$ and integrate from $x\geqq\beta$ to $t\geqq x$ to get

$$\frac{y'(t)}{f(y(t))} - \frac{y'(x)}{f(y(x))} + \int_x^t f'(y(s))\left[\frac{y'(s)}{f(y(s))}\right]^2 ds + \int_x^t a(s)ds = 0. \tag{3}$$

The first integral is nonnegative; we may suppose β so large that the second integral is positive for t sufficiently large. Thus

$$y'(t)/f(y(t)) < y'(x)/f(y(x)) \tag{4}$$

provided t is sufficiently larger than x. We distinguish two cases.

Case 1. $y(x)y'(x)\leqq 0$ for some $x\geqq\beta$. Then the inequality above shows that $y(t)y'(t)<0$ for all sufficiently large t, so $|y|$ is monotone decreasing. Moreover, (4) implies that there is no loss of generality in assuming $y'/f(y)<0$ on $[\beta, \infty)$. If $|y(t)|\geqq L>0$, then $f(y(t))$ is bounded away from zero, say $|f(y(t))|\geqq\delta$, whence (4) yields

$$|y'(t)| > \left|\frac{y'(x)}{f(y(x))}\right|\delta.$$

But $yy'<0$, y' bounded away from zero is incompatible with the fact that y does not vanish on $[x, \infty)$. Thus y tends monotonically to zero.

Case 2. $y(x)y'(x)>0$ for $x\in[\beta, \infty)$. In this case y is monotone increasing and y is bounded away from zero. From

$$\limsup_{t\to\infty}\int_\beta^t \frac{y'(s)}{f(y(s))}\,ds = \limsup_{t\to\infty}\int_{y(\beta)}^{y(t)} \frac{du}{f(u)} < \infty$$

we conclude that $\liminf y'(s)/f(y(s))=0$ whence, in view of (4), $y'(s)/f(y(s))\to 0$ as $s\to+\infty$. Letting $t\to+\infty$ in (3), we obtain the inequality

$$\frac{y'(x)}{f(y(x))} \geqq \lim_{t\to\infty}\int_x^t a(s)ds. \tag{5}$$

If the indicated limit is $+\infty$ for some $x\in[\beta,\infty)$, we have a contradiction to the continuity of y', since $y(x)\neq 0$. Suppose then that the limit exists for all large x and integrate (5) from x to $X\geqq x$, obtaining

$$\int_{y(x)}^{y(X)}\frac{du}{f(u)} \geqq \int_x^X sa(s)ds - x\int_x^X a(s)ds + (X-x)\int_X^\infty a(s)ds.$$

The second integral on the right is bounded in X and the third is nonnegative; the first diverges to $+\infty$ as $X\to\infty$. Since the left side is bounded, we have a contradiction. Thus Case 2 is impossible, and the theorem is proved.

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that*

$$\lim_{\epsilon\to 0+}\int_\epsilon^1 \frac{du}{f(u)} < \infty, \qquad \lim_{\epsilon\to 0+}\int_{-\epsilon}^{-1}\frac{du}{f(u)} < \infty.$$

Then every $y\in\mathfrak{F}$ *is oscillatory.*

PROOF. Proceeding as above, we have only to show that Case 1 of the proof of Theorem 1 now leads to a contradiction. Thus we assume that $y(t)y'(t)<0$, $y(t)\to 0$. From

$$\limsup_{t\to\infty}\int_\beta^t \left|\frac{y'(s)}{f(y(s))}\right| ds = \limsup_{y\to 0}\int_y^{y(\beta)}\frac{du}{f(u)} < \infty,$$

we have

$$\liminf_{t\to\infty}\left|\frac{y'(s)}{f(y(s))}\right| = 0,$$

so there exists a sequence (s_k), $s_k\to\infty$, such that $y'(s_k)/f(y(s_k))\to 0$ as $k\to\infty$. Let $t=s_k$ in (3) and take the limit as $k\to\infty$ to get

$$\frac{y'(x)}{f(y(x))} \geqq \lim_{k\to\infty}\int_x^{s_k} a(s)ds > 0,$$

a contradiction.

The function $f(u)=|u|^{1/2}\operatorname{sgn} u+u^3$ is an example of a function f satisfying the conditions of Theorem 2. By strengthening the assumptions on a slightly, we can extend the class of functions f to include $u+u^3$, for example. In the following we denote $\min\{a(t), 0\}$ by $a_-(t)$.

THEOREM 3. *In addition to the hypotheses of Theorem* 1, *assume that* $f'\geqq\gamma>0$ *on* $(-\infty, 0)\cup(0, \infty)$ *and that* $a_-(s)$ *is integrable over* $(0, \infty)$. *Then every* $y\in\mathfrak{F}$ *is oscillatory.*

PROOF. We again show that Case 1 of Theorem 1 is impossible. Assume then that $y>0$, $y'<0$ on $[\beta, \infty)$; a similar argument treats the case $y<0$, $y'>0$. From (4) we have

$$\frac{y'(t)}{f(y(t))}\leqq-\delta<0$$

for some δ; it follows that the first integral in (3) diverges to $+\infty$ as $t\to\infty$. Hence $y'(t)/f(y(t))\to-\infty$ as $t\to\infty$.

Integration of (1) leads to the inequality

$$y'(t)-y'(x)+\left[\sup_{x\leqq s\leqq t} f(y(s))\right]\int_x^t a_-(s)ds\leqq 0$$

for $t\geqq x$, whence

$$y'(x)\geqq y'(t)-Mf(y(x)),$$

using the monotonicity of f; here $-M=\int_0^\infty a_-(s)ds$. Now $\limsup_{t\to\infty} y'(t)=0$ since $y(t)>0$, so we may take lim sup in the above inequality to obtain

$$y'(x)/f(y(x))\geqq-M,$$

a contradiction. Thus every solution of (1) defined on a positive half-line is oscillatory.

The function $a(t)=(1/t)\cos t+3/t^2$ satisfies the conditions of Theorem 2 but not those of Theorem 3.

The theorems proved here can be extended to a somewhat larger class of equations by considerations of the sort used in [5].

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UNIVERSITY OF IDAHO