APPROXIMATE EVALUATION OF A CLASS OF WIENER INTEGRALS

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1. It is well known that for every $\alpha \in (0, \frac{1}{2})$ almost every element $x$ of the Wiener space $C_w$ consisting of all the real valued functions $x(t), t \in [0, 1]$, with $x(0) = 0$ satisfies the Hölder condition $|x(t') - x(t'')| \leq h|t' - t''|^{\alpha}$ for some $h > 0$ which depends on $x$ and $\alpha$. Let $\phi_{a}[x]$ be the infimum of all such $h > 0$ for fixed $x$ and $\alpha$. In [7] we showed that for every $\alpha \in (0, \frac{1}{2})$ the functional $\phi_{a}[x]$ is Wiener integrable for every $p \geq 0$ and in fact

$$\int_{C_w} \phi_{a}^{p}[x] d\mu[x] \leq 2^{p}(1 - 2^{-\alpha})^{-p} \cdot \sum_{m=1}^{\infty} (m + 1)^{p}m^{-(m+1)} < \infty.$$ 

We then applied this result to estimate errors in approximating Wiener integrals of a class of functionals by Lebesgue integrals in Euclidean spaces. In the present paper we apply this method to yet another class of Wiener integrals. Our result is the following:

**Theorem.** Let $f(t)$ be real valued for $t \in [0, 1]$ and satisfy

(1) $\left| f(t') - f(t'') \right| \leq C |t' - t''|^\gamma$

where $C, \gamma > 0$ and let $A = \max_{[0,1]} |f(t)|$. Let $g(u)$ be real valued for all real $u$ and satisfy

(2) $\left| g(u') - g(u'') \right| \leq B |u' - u''|

with $B > 0$. Then the functionals $F[x]$ and $F_{n}[x]$ defined on $C_w$ by

(3) $F[x] = g \left\{ \int_{0}^{1} f(t)x^{2}(t)dt \right\}$

(4) $F_{n}[x] = g \left\{ \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right)x^{2} \left( \frac{k}{n} \right) \right\}$, $n = 1, 2, \cdots$

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are Wiener measurable, and for \( \alpha \in (0, \frac{1}{6}) \)

\[
\left| \int_{C_w} F[x] d_w x - \int_{C_w} F_n[x] d_w x \right| \\
\leq \frac{2BC}{(\gamma + 1)n^\gamma} + \frac{2^{3/2}AB}{(\alpha + 1)n^\alpha} \left\{ \int_{C_w} \phi_\alpha^2[x] d_w x \right\}^{1/2}.
\]

The Wiener integral \( \int_{C_w} F_n[x] d_w x \) which approximates \( \int_{C_w} F[x] d_w x \) can be evaluated as a Lebesgue integral in the \( n \)-dimensional Euclidean space according to the well-known fact that if \( G(\xi_1, \ldots, \xi_n) \) is a Lebesgue measurable function defined on the \( n \)-dimensional Euclidean space and \( 0 = t_0 < t_1 < \cdots < t_n \leq 1 \) then \( G[x(t_1), \ldots, x(t_n)] \) is Wiener measurable and furthermore

\[
\int_{C_w} G[x(t_1), \cdots, x(t_n)] d_w x = \left\{ (2\pi)^n \sum_{k=1}^n (t_k - t_{k-1}) \right\}^{-1/2}
\]

\[
\left. \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty G(\xi_1, \cdots, \xi_n) \exp \left\{ - \sum_{k=1}^n \frac{(\xi_k - \xi_{k-1})^2}{2(t_k - t_{k-1})} \right\} d\xi_1 \cdots d\xi_n \right|\text{ in the sense that the existence of one side implies that of the other and the equality of the two.}
\]

Thus for our \( F_n[x] \) we have

\[
\int_{C_w} F_n[x] d_w x = (2\pi)^{-n/2} n^{n/2} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \left\{ \frac{1}{n} \sum_{k=1}^n f \left( \frac{k}{n} \right) \xi_k \right\}^2 \exp \left\{ - \frac{n}{2} \sum_{k=1}^n (\xi_k - \xi_{k-1})^2 \right\} d\xi_1 \cdots d\xi_n.
\]

2. In proving our theorem we need to estimate the Wiener integral

\[
\int_{C_w} |||x|||^2 d_w x \quad \text{where } |||x||| = \max_{[0,1]} |x(t)|.
\]

In [2] P. Erdös and M. Kac showed that if \( X_1, X_2, X_3, \cdots \) are independent identically distributed random variables each having mean value 0 and standard deviation 1 and if \( s_k = X_1 + X_2 + \cdots + X_k \) then

\[
\lim_{n \to \infty} \text{prob.}\left\{ \max(s_1, s_2, \cdots, s_n) < \alpha n^{1/2} \right\} = \sigma(\alpha)
\]

where \( \sigma(\alpha) = 0 \) for \( \alpha \leq 0 \) and

\[
\sigma(\alpha) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\alpha \exp \left\{ - \frac{u^2}{2} \right\} du \quad \text{for } \alpha \geq 0.
\]

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From this follows immediately

**Theorem A.** Let \( f(u) \) be measurable on \([0, \infty)\). Then if either of the following integrals exists, both exist and they are equal:

\[
\int_{c_w} f(\max x(t)) d\omega x = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{ -\frac{u^2}{2} \right\} du.
\]

**Corollary B.** Let \( f(u) \) be measurable and nonnegative on \([0, \infty)\). Then

\[
\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{ -\frac{u^2}{2} \right\} du \leq \int_{c_w} f(||x||) d\omega x
\]

\[
\leq 2 \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{ -\frac{u^2}{2} \right\} du
\]

where the left-hand inequality holds provided that \( f(u) \) is monotonically increasing.

Corollary B follows from Theorem A and the observation that

\[
|||x||| = \max\{u, -u\}
\]

so that

\[
|||x||| = \max\{\max x(t), \max [-x(t)]\}
\]

and hence if \( f(u) \geq 0 \) (and for the left-hand inequality, \( f \) is also monotonically increasing),

\[
f(\max x(t)) \leq f(|||x|||) \leq f(\max x(t)) + f(\max [-x(t)])
\]

so that finally

\[
\int_{c_w} f(\max x(t)) d\omega x \leq \int_{c_w} f(|||x|||) d\omega x
\]

\[
\leq \int_{c_w} f(\max x(t)) d\omega x + \int_{c_w} f(\max [-x(t)]) d\omega x
\]

\[
= 2 \int_{c_w} f(\max x(t)) d\omega x.
\]

**3. Proof of the theorem.** From the continuity of \( g(u) \), the function

\[
g \left\{ \frac{1}{n} \sum_{k=1}^n \left( \frac{k}{n} \right) \xi_k \right\}
\]

is continuous on the \( n \)-dimensional Euclidean space with elements \((\xi_1, \cdots, \xi_n)\) and hence \( F_n [x] \) as defined by (4) is Wiener measurable.
Since \( x(t) \) and \( f(t) \) are continuous the Riemann sum in (4) converges to the Riemann integral in (3) as \( n \to \infty \). Then from the continuity of \( g(u) \), 
\[
\lim_{n \to \infty} F_n[x] = F[x]
\]
for every \( x \in C_w \). This establishes the Wiener measurability of \( F[x] \). Now 
\[
\left| \int_0^1 f(t)x^2(t)dt - \frac{1}{n} \sum_{k=1}^n f\left( \frac{k}{n} \right) x^2\left( \frac{k}{n} \right) \right|
\]

\[
\leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t)x^2(t) - f\left( \frac{k}{n} \right) x^2\left( \frac{k}{n} \right) \right| dt
\]

\[
\leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t)x^2(t) - f\left( \frac{k}{n} \right) x^2(t) \right| dt
\]

\[
+ \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f\left( \frac{k}{n} \right) x^2(t) - f\left( \frac{k}{n} \right) x^2\left( \frac{k}{n} \right) \right| dt
\]

\[
\leq \|x\|^2 \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t) - f\left( \frac{k}{n} \right) \right| dt
\]

\[
+ 2A\|x\| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| x(t) - x\left( \frac{k}{n} \right) \right| dt
\]

\[
\leq C\|x\|^2 \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left( \frac{k}{n} - t \right)^\gamma dt
\]

\[
+ 2A\|x\| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \phi_\alpha[x] \left( \frac{k}{n} - t \right)^\alpha dt
\]

\[
\leq \frac{C}{(\gamma + 1)n^\gamma} \|x\|^2 + \frac{2A}{(\alpha + 1)n^\alpha} \|x\| \phi_\alpha[x].
\]

Thus by (2) 
\[
\left| F[x] - F_n[x] \right| = \left| g \left\{ \int_0^1 f(t)x^2(t)dt \right\} - g \left\{ \frac{1}{n} \sum_{k=1}^n f\left( \frac{k}{n} \right) x^2\left( \frac{k}{n} \right) \right\} \right|
\]

\[
\leq B \left| \int_0^1 f(t)x^2(t)dt - \frac{1}{n} \sum_{k=1}^n f\left( \frac{k}{n} \right) x^2\left( \frac{k}{n} \right) \right|
\]

\[
\leq \frac{BC}{(\gamma + 1)n^\gamma} \|x\|^2 + \frac{2AB}{(\alpha + 1)n^\alpha} \|x\| \phi_\alpha[x]
\]

and hence
\[ \left| \int_{c_w} F[x]d\omega x - \int_{c_w} F_n[x]d\omega x \right| \leq \frac{BC}{(\gamma + 1)n^\gamma} \int_{c_w} |||x|||^2d\omega x + \frac{2AB}{(\alpha + 1)n^\alpha} \left\{ \int_{c_w} |||x|||^2d\omega x \right\}^{1/2} \left\{ \int_{c_w} \phi_n^2[x]d\omega x \right\}^{1/2}. \]

Finally according to Corollary B

\[ \int_{c_w} |||x|||^2d\omega x \leq 2\left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} u^2 \exp\left\{ -\frac{u^2}{2} \right\} du = 2. \]

This completes the proof of (5).

**BIBLIOGRAPHY**


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