EQUIVALENT METRICS GIVING DIFFERENT VALUES TO METRIC-DEPENDENT DIMENSION FUNCTIONS[1]

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In [1] K. Nagami and J. H. Roberts introduced metric-dependent dimension functions d_2 and d_3 defined on the class of all metric spaces. All definitions are given below. The definition of d_5 is due to R. E. Hodel [5]. The following relations hold for all metric spaces (X, ρ):

$$d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_5(X, \rho) \leqq \mu\dim(X, \rho) \leqq \dim X,$$

where μdim is metric dimension as defined by Katětov [3], and dim X is covering dimension.

The following is a natural question. Suppose d is d_2, d_3, d_5, or μdim; and suppose $d(X, \rho) = r < n = \dim X$. Then for every k $(r \leqq k \leqq n)$ does there exist a topologically equivalent metric ρ_k for X such that $d(X, \rho_k) = k$? Roberts and Slaughter [2] answered this question in the affirmative when d is μdim. Roberts [6] answered this question in the affirmative for all separable metric spaces when d is d_3. This paper answers this question in the affirmative for all metric spaces when d is d_3 or d_5. The question remains unanswered when d is d_2.

In the following if S is a set, $|S|$ will denote the cardinality of S.

Definition. Let η be any ordinal number. A metric space (X, ρ) is said to have property $P(|\eta|, k, \rho)$ if given any collection of pairs of closed sets indexed by η, $\mathcal{C} = \{(C_\alpha, C_\alpha'): \alpha < \eta\}$ such that there exists an $\epsilon > 0$ with $\rho(C_\alpha, C_\alpha') \geqq \epsilon$ for all $\alpha < \eta$ then there exists a collection of closed sets $\{B_\alpha : \alpha < \eta\}$ such that B_α separates X between C_α and C_α' and order $\{B_\alpha : \alpha < \eta\} \leqq k$.

Definition. $d_2(X, \rho)$ is the smallest integer n such that (X, ρ) has property $P(n+1, n, \rho)$.

Definition. $d_3(X, \rho)$ is the smallest integer n such that (X, ρ) has property $P(m, n, \rho)$ for every integer m.

Definition. $d_5(X, \rho)$ is the smallest integer n such that (X, ρ) has property $P(\aleph_0, n, \rho)$.

Definition. μdim (X, ρ) is the smallest integer n such that for each $\epsilon > 0$ there exists an open cover $\mathcal{U}$ of X with

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(i) ρ-mesh $\mathfrak{U} \leqq \epsilon$ and
(ii) order $\mathfrak{U} \leqq n+1$.

LEMMA. *Suppose (X, ρ) is a metric space, $f: X \to [0, 1]$ is a continuous function and $\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|$. Then σ is a metric on X topologically equivalent to ρ. (See* [**4**, p. 199].)

THEOREM 1. *Suppose (X, ρ) is a metric space, $f: X \to [0, 1]$ a continuous function and $\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|$.*

(1) *If η is any ordinal number such that $\aleph_0 \leqq |\eta| \leqq 2^{\aleph_0}$ and (X, ρ) has property $P(|\eta|, k, \rho)$ then (X, σ) has property $P(|\eta|, k+1, \sigma)$.*

(2) If *(X, ρ) has property $P(m, k, \rho)$ for every integer m then (X, σ) has property $P(m, k+1, \sigma)$ for every integer m.*

REMARK. To prove Theorem 2, the principle result of this paper, Theorem 1 is used only for the cases where η is countable or finite.

PROOF OF THEOREM 1. Let $\mathfrak{C} = \{(C_\alpha, C_\alpha'): \alpha < \eta\}$ be any collection of pairs of closed sets with $|\eta| \geqq 2^{\aleph_0}$ and with $\sigma(C_\alpha, C_\alpha') \geqq \epsilon$ for all $\alpha < \eta$ and for some $\epsilon > 0$. Choose an integer N_0 so that $1/N_0 < \epsilon/4$. Since $|\eta| \leqq 2^{\aleph_0}$ there exists a set T_0 with $T_0 \subseteq [0, 1/N_0)$ and with $|T_0| = |\eta|$. We can assume that $T_0 = \{t_\alpha: \alpha < \eta\}$ where if $t_\alpha = t_\beta$ then $\alpha = \beta$. Let $S = \{0, 1, 2, \cdots, N_0\}$ and let $t_\alpha^0 = 0$ and $t_\alpha^{N_0+1} = 1$ for all $\alpha < \eta$. For each i, $1 \leqq i \leqq N_0$ and each $\alpha < \eta$ let $t_\alpha^i = t_\alpha + (i-1)/N_0$. Now for each $i \in S$ and each $\alpha < \eta$ define

$$E_\alpha^i = \{x: t_\alpha^i \leqq f(x) \leqq t_\alpha^{i+1}\} \text{ and define}$$

$$D_\alpha^i = \{x: \rho((C_\alpha \cap E_\alpha^i), x) \geqq \epsilon/4\}.$$

Then for $i \in S$ and $\alpha < \eta$ we have $\rho(C_\alpha \cap E_\alpha^i, C_\alpha' \cap E_\alpha^i) \geqq 3\epsilon/4$ because for $x, y \in E_\alpha^i$ we have $|f(x) - f(y)| \leqq 1/N_0 \leqq \epsilon/4$. Thus $D_\alpha^i \supseteq C_\alpha' \cap E_\alpha^i$ for all $i \in S$ and $\alpha < \eta$. Now $\mathfrak{D} = \{(D_\alpha^i, C_\alpha \cap E_\alpha^i): i \in S, \alpha < \eta\}$ is a collection of pairs of closed sets with $\rho(D_\alpha^i, C_\alpha \cap E_\alpha^i) \geqq \epsilon/4$. In Case (1) where $\aleph_0 \leqq |\eta| \leqq 2^{\aleph_0}$, we have $|\mathfrak{D}| = |\eta|$. In Case (2) we have $|\eta| = m$ for some integer m and $|\mathfrak{D}| = mN_0$. Thus in either case our hypothesis guarantees the existence of a collection of closed sets $\{B_\alpha^i: i \in S, \alpha < \eta\}$ with

(i) order $\{B_\alpha^i: i \in S, \alpha < \eta\} \leqq k$ and

(ii) $X - B_\alpha^i = U_\alpha^i \cup V_\alpha^i$ where U_α^i and V_α^i are disjoint open sets and $D_\alpha^i \subseteq V_\alpha^i$ and $C_\alpha \cap E_\alpha^i \subseteq U_\alpha^i$.

For each $\alpha < \eta$ we will modify the collection $\{B_\alpha^i: i \in S\}$ to obtain a closed set B_α separating X between C_α and C_α' using a variation of a method due to J. H. Roberts [7].

For each $i \in S$ and $\alpha < \eta$ define $L_\alpha^i = \{x: f(x) = t_\alpha^i\}$. Notice that

$L_\alpha^i = E_\alpha^{i-1} \cap E_\alpha^i$, for i such that $1 \leqq i \leqq N_0$. For each $\alpha < \eta$ let $L_\alpha^{N_0+1} = \varnothing$ and define

$$B_\alpha = \bigcup_{j=0}^{N_0} \left[(B_\alpha^j \cap E_\alpha^j) \cup (L_\alpha^{j+1} \cap [(U_\alpha^j - U_\alpha^{j+1}) \cup (U_\alpha^{j+1} - U_\alpha^j)])\right],$$

$$U_\alpha = \bigcup_{j=0}^{N_0} (U_\alpha^j \cap E_\alpha^j) - B_\alpha,$$

$$V_\alpha = \bigcup_{j=0}^{N_0} (V_\alpha^j \cap E_\alpha^j) - B_\alpha.$$

Assertion 1. For each $\alpha < \eta$ B_α is a closed set separating X between C_α and C'_α.

Proof. First we show that B_α is closed. Let $H_\alpha = \bigcup_{j=0}^{N_0} (B_\alpha^j \cap E_\alpha^j)$ and let $G_\alpha = \bigcup_{j=0}^{N_0} (L_\alpha^{j+1} \cap [(U_\alpha^j - U_\alpha^{j+1}) \cup (U_\alpha^{j+1} - U_\alpha^j)])$. It suffices to show that $\overline{G}_\alpha \subseteq B_\alpha$ since H_α is closed. If x is a limit point of G_α then there exists some $k \in S$ such that x is a limit point of $L_\alpha^{k+1} \cap [(U_\alpha^k - U_\alpha^{k+1}) \cup (U_\alpha^{k+1} - U_\alpha^k)]$. We may assume then that x is a limit point of $(U_\alpha^k - U_\alpha^{k+1})$ hence a limit point of U_α^k. But $X - B_\alpha^k = U_\alpha^k \cup V_\alpha^k$ where $U_\alpha^k \cap V_\alpha^k = \varnothing$. Thus either $x \in U_\alpha^k$ or $x \in B_\alpha^k$ and in either case $x \in B_\alpha$ so B_α is closed.

Next we show that $X - B_\alpha = U_\alpha \cup V_\alpha$. If $x \in X$, there exists $k \in S$ such that $x \in E_\alpha^k$, since $\bigcup_{j=0}^{N_0} E_\alpha^j = X$. If $x \notin B_\alpha$ then surely $x \notin B_\alpha^k \cap E_\alpha^k$. But $X - B_\alpha^k = U_\alpha^k \cup V_\alpha^k$ so x is in one of U_α^k or V_α^k hence one of U_α or V_α.

We show that $U_\alpha \cap V_\alpha = \varnothing$. If $x \in U_\alpha$ then either $x \in E_\alpha^k$ for exactly one $k \in S$ or $x \in (E_\alpha^k \cap E_\alpha^{k+1})$ for exactly one $k \in S$. In the first case since $x \in U_\alpha^k$ we have $x \notin V_\alpha^k$ hence $x \notin V_\alpha$. In the second case we can suppose that $x \in U_\alpha^k$. The only possibility to have $x \in V_\alpha$ is to have $x \in V_\alpha^{k+1}$. But then $x \notin U_\alpha^{k+1}$ hence $x \in (U_\alpha^k - U_\alpha^{k+1}) \cap (E_\alpha^k \cap E_\alpha^{k+1})$. Thus $x \in B_\alpha$ and $x \notin V_\alpha$ so we conclude that $U_\alpha \cap V_\alpha = \varnothing$.

To show that $C_\alpha \subseteq U_\alpha$ we first show that $C_\alpha \cap B_\alpha = \varnothing$. Let $x \in C_\alpha$ and suppose that $x \in E_\alpha^k$ for exactly one k. Then $x \notin L_\alpha^j$ for any $j \in S$. Now $(C_\alpha \cap E_\alpha^k) \cap B_\alpha^k = \varnothing$ because $X - B_\alpha^k = U_\alpha^k \cup V_\alpha^k$ where $(C_\alpha \cap E_\alpha^k) \subseteq U_\alpha^k$. So in this case $x \notin B_\alpha$. If $x \in E_\alpha^k \cap E_\alpha^{k+1}$ for some $k \in S$ then $x \in U_\alpha^k$ and $x \in U_\alpha^{k+1}$. Thus $x \notin B_\alpha$ and $C_\alpha \cap B_\alpha = \varnothing$. Since $C_\alpha \subseteq \bigcup_{j=0}^{N_0} (U_\alpha^j \cap E_\alpha^j)$ and $C_\alpha \cap B_\alpha = \varnothing$ we conclude that $C_\alpha \subseteq U_\alpha$.

From the definition of D_α^j it is clear that $C'_\alpha \subseteq \bigcup_{j=0}^{N_0} D_\alpha^j$ but $D_\alpha^j \subseteq V_\alpha^j$ so $C'_\alpha \subseteq \bigcup_{j=0}^{N_0} V_\alpha^j$. Thus if we show that $C'_\alpha \cap B_\alpha = \varnothing$ we can conclude that $C'_\alpha \subseteq V_\alpha$. Let $x \in C'_\alpha$ and let $x \in E_\alpha^k$ for exactly one k. Then $x \in V_\alpha^k$ so $x \notin B_\alpha^k$ hence $x \notin B_\alpha$. If x is in $E_\alpha^k \cap E_\alpha^{k+1}$ then $x \in V_\alpha^k$ and $x \in V_\alpha^{k+1}$ hence $x \notin U_\alpha^k$ and $x \notin U_\alpha^{k+1}$ so $x \notin B_\alpha$. Thus $C'_\alpha \subseteq V_\alpha$.

We show that U_α is open. Let $x \in U_\alpha$. Then $x \notin B_\alpha$ a closed set so there exists an open set M_x containing x with $M_x \cap B_\alpha = \emptyset$. Suppose $x \in E_\alpha^k$ for some unique k. Then x is in the interior of E_α^k so there exists an open set N_x with $x \in N_x \subseteq E_\alpha^k$. Then $x \in M_x \cap N_x \cap U_\alpha^k \subseteq U_\alpha$. Suppose $x \in L_\alpha^{k+1}$ some $k \in S$. Since x is in the interior of $E_\alpha^k \cup E_\alpha^{k+1}$ choose an open set N_x so that $x \in N_x \subseteq E_\alpha^k \cup E_\alpha^{k+1}$. Since $x \in L_\alpha^{k+1}$ and $x \in U_\alpha^k$ and $x \notin B_\alpha$ we have $x \in U_\alpha^{k+1}$. Thus $x \in (U_\alpha^k \cap U_\alpha^{k+1}) \cap M_x \cap N_x \subseteq U_\alpha$. Thus U_α is open. A similar argument shows that V_α is open. This completes the proof of Assertion 1.

ASSERTION 2. Order $\{B_\alpha : \alpha < \eta\} \leqq k+1$.

PROOF. Let $P_\alpha = \bigcup_{j=0}^{N_0} (B_\alpha^j \cap E_\alpha^j)$. Then order $\{P_\alpha : \alpha < \eta\} \leqq k$ since order $\{B_\alpha^j : j \in S,\ \alpha < \eta\} \leqq k$. For $\alpha < \eta$ let $Q_\alpha = \bigcup_{j=1}^{N_0} L_\alpha^j$. Now order $\{Q_\alpha : \alpha < \eta\} \leqq 1$. Hence order $\{B_\alpha : \alpha < \eta\} \leqq$ order $\{(P_\alpha \cap Q_\alpha) : \alpha < \eta\} \leqq k+1$. This completes the proof of the theorem.

COROLLARY. *Let (X, ρ) be a metric space, $f: X \rightarrow [0, 1]$ a continuous function, $\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|$, and let d be d_3 or d_5. If $d(X, \rho) \leqq k$ then $k \leqq d(X, \sigma) \leqq k+1$.*

THEOREM 2. *Let (X, ρ) be a metric space and let d be d_3 or d_5. Suppose $d(X, \rho) = r < n = \dim X$. Then for each k, $r \leqq k \leqq n$ there exists a topologically equivalent metric ρ_k for X such that $d(X, \rho_k) = k$.*

PROOF. Let $C_1, C_1'; C_2, C_2'; \cdots, C_n, C_n'$ be n pairs of disjoint closed sets with the property that if for each $i = 1, \cdots, n$ B_i is a closed set separating C_i and C_i', then $\bigcap_{i=1}^n B_i \neq \emptyset$. This is possible since $\dim X = n$. For each $i = 1, \cdots, n$ let $f_i: X \rightarrow [0, 1]$ such that f is continuous, $f_i(C_i) = 0$ and $f_i(C_i') = 1$. For each $i = 1, \cdots, n$ define $\sigma_i: X \times X \rightarrow$ real numbers by

$$\sigma_i(x, y) = \rho(x, y) + \sum_{j=1}^{i} |f_j(x) - f_j(y)|.$$

Now $\sigma_n(C_i, C_i') \geqq 1$ for all $i = 1, \cdots, n$ thus $d(X, \sigma_n) \geqq n$. But by the above corollary $d(X, \sigma_{i+1}) \leqq d(X, \sigma_i) + 1$. Thus all values k, $r \leqq k \leqq n$, are assumed and the theorem is proved.

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