GENERATING FUNCTIONS FOR JACOBI AND LAGUERRE POLYNOMIALS

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Let \( v \) be a function of \( t \) defined by

\[
(1) \quad v = t(1 + v)^{b+1}, \quad v(0) = 0.
\]

Then it follows from Lagrange’s expansion formula [6, Vol. I, p. 126, Ex. 212] that

\[
(2) \quad \frac{(1 + v)^{a+1}}{1 - bv} = \sum_{n=0}^{\infty} \left( a + (b + 1)n \right) \frac{t^n}{n!}.
\]

Making use of the formula (2), Carlitz [2] has proved that the Laguerre polynomial \( L_n^{(a+b,n)}(x) \), where

\[
(3) \quad L_n^{(a)}(x) = \sum_{k=0}^{n} (-1)^k \binom{a+n}{n-k} \frac{x^k}{k!},
\]

satisfies a generating relation in the form

\[
(4) \quad \sum_{n=0}^{\infty} L_n^{(a+b,n)}(x) t^n = \frac{(1 + v)^{a+1}}{1 - bv} \exp(-xv),
\]

where \( v \) is given by (1) and \( a, b \) are arbitrary complex numbers. Note that the special case of (4) when \( b \) is an arbitrary integer was proved earlier by Brown [1].

In terms of the generalized hypergeometric function

\[
(5) \quad {}_pF_q \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} \right] x = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{x^n}{n!},
\]

where

\[
(6) \quad (\lambda)_n = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), \quad n \geq 1, \quad (\lambda)_0 = 1,
\]

the generating relation (4) assumes the form
\[
\sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} \frac{-n;}{1 + a + bn; x} t^n
\]
\[
= \frac{(1 + v)^{a+1}}{1 - bv} \exp(-xv).
\]

In (7) if we replace \(x\) by \(xz\), multiply both sides by \(z^\lambda-1\) and take their Laplace transforms with respect to the variable \(z\), we shall readily obtain

\[
\sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} \frac{-n, \lambda;}{1 + a + bn; x} t^n
\]
\[
= \frac{(1 + v)^{a+1}}{1 - bv} (1 + xv)^{-\lambda},
\]

where the binomial \((1+\lambda v)^{-\lambda}\) may be written as an \(1F_0\).

The form of (8) suggests the existence of the general formula

\[
\sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} \frac{-n, \alpha_1, \cdots, \alpha_p;}{1 + a + bn, \beta_1, \cdots, \beta_q; x} t^n
\]
\[
= \frac{(1 + v)^{a+1}}{1 - bv} \frac{pF_q}{pF_q} \left[ \begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \\ -xv \end{array} \right],
\]

where \(p, q\) are nonnegative integers, the \(\alpha\)'s and \(a, b\) take general values, real or complex, and

\[
\beta_j \neq 0, -1, -2, \cdots, \quad j = 1, 2, \cdots, q.
\]

The derivation of (9) from (7) and (8) by the principle of multi-dimensional mathematical induction would require the Laplace and inverse Laplace transform techniques illustrated, for instance, by the author [7].

For a direct proof without using (7) and (8) we notice that, in view of the definition (5),

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\[ \sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} F_{q+1}^{p+1} \left[ \begin{array}{c} -n, \alpha_1, \ldots, \alpha_p; \\ 1 + a + bn, \beta_1, \ldots, \beta_q; \end{array} \right] t^n \]

\[ = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \binom{a + (b + 1)n}{n - k} \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} \frac{x^k}{k!} \]

\[ = \sum_{k=0}^{\infty} (-1)^k \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} \frac{x^k}{k!} \sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n - k} t^n \]

\[ = \sum_{k=0}^{\infty} (-1)^k \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} \frac{x^k t^k}{k!} \frac{(1 + v)^{a + (b + 1)k + 1}}{1 - bv}, \]

by (2), and the formula (9) follows immediately. We can easily attribute a direct proof to the formula (8) which obviously corresponds to the special case \( p = 1, q = 0 \) of (9).

A similar generalization of Carlitz's formula \([2, \text{p. 827, Equation (16)}]\) has the form

\[ \sum_{n=0}^{\infty} \binom{-a - bn}{n} F_{q+1}^{p+1} \left[ \begin{array}{c} -n, \alpha_1, \ldots, \alpha_p; \\ 1 - a - (b + 1)n, \beta_1, \ldots, \beta_q; \end{array} \right] t^n \]

(11)

\[ = \frac{A(-t, a, b)}{1 - B(-t, b)} F_q^{p} \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} \right] \frac{xB(-t, b)}{1 - B(-t, b)}, \]

where, for convenience,

(12) \[ B(t, b) = -\sum_{n=1}^{\infty} \binom{(b + 1)n}{n - 1} \frac{t^n}{n} \]

and

(13) \[ A(t, a, b) = \frac{(1 - B(t, b))^{a+1}}{1 + bB(t, b)}. \]

Indeed the formula (11) is obtainable from (9) by replacing \( a \) by \(-a\) and \( b \) by \(-(b+1)\).
It may be of interest to remark that for \( b = 0 \) and \( b = -1 \) the formula (9) yields Chaundy's results (25) and (27) respectively (see [4, p. 62]). For \( b = -\frac{1}{2} \), (9) reduces to the generating relation (7), p. 264 of Brown's recent paper.²

For the Jacobi polynomial defined by

\[
P_{n}^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \binom{\alpha + n}{k} \binom{\beta + n}{n - k} \left( \frac{x - 1}{2} \right)^{n-k} \left( \frac{x + 1}{2} \right)^{k},
\]

it is easy to show from the identity (4.22.1) of [8, p. 63] that

\[
P_{n}^{(\alpha - n, \beta - n)}(x) = \binom{n - \alpha - \beta - 1}{n} (1 - x)^n \frac{2}{1-x} _2F_1 \left[ \begin{array}{c} -n, -\alpha; \\ -\alpha - \beta; \end{array} \right],
\]

and therefore (8) gives us the elegant generating function

\[
\sum_{n=0}^{\infty} P_{n}^{(\alpha-n, \beta-(b+1)n)}(x) t^n = (1 + w)^{-\alpha-\beta} (1 + bw)^{-1} \left( 1 + \frac{2w}{1-x} \right)^{\alpha},
\]

where

\[
w = \frac{1}{2} (1 - x) (1 + w)^{b+1}.
\]

Evidently (16) reduces to the known formula [3, p. 88]

\[
\sum_{n=0}^{\infty} P_{n}^{(\alpha-n, \beta-n)}(x) t^n = [1 + \frac{1}{2}(x + 1) t]^{\alpha} [1 + \frac{1}{2}(x - 1) t]^{\beta}
\]

when \( b = 0 \), and for \( b = -1 \) it leads us to Feldheim's result [5, p.120]

\[
\sum_{n=0}^{\infty} P_{n}^{(\alpha-n, \beta-n)}(x) t^n = (1 + t)^{\alpha} [1 - \frac{1}{2}(x - 1) t]^{-\alpha-\beta-1}.
\]

Now from the definition (14) we readily have [8, p. 61]

\[
P_{n}^{(\alpha, \beta)}(x) = \binom{\alpha + n}{n} _2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \end{array} \right],
\]

whence it follows at once that

Consequently, \( (8) \) gives us another class of generating functions for the Jacobi polynomial in the form

\[
\sum_{n=0}^{\infty} P_n^{(a+bn, \beta-(b+1)n)}(x) t^n = (1 + v)^{a+1} (1 - bv)^{-1} [1 - (x - 1)v]^{-a - \beta - 1},
\]

where \( v \) is defined by \( (1) \) and \( b, \alpha, \beta \) are unrestricted, in general.

For \( b = -1 \), \( (22) \) leads us again to Feldheim's formula \( (19) \); when \( b = 0 \), it reduces to the generating relation

\[
\sum_{n=0}^{\infty} P_n^{(a, \beta-n)}(x) t^n = (1 - t)^{\beta} [1 - \frac{1}{2}(x + 1)t]^{-a - \beta - 1}
\]

also due to Feldheim [5, p. 120].

Finally, we remark that the special case \( b = - \frac{1}{2} \) of our formula \( (22) \) corresponds to

\[
\sum_{n=0}^{\infty} P_n^{(a-n/2, \beta-n/2)}(x) t^n = [1 + u(t)]^{a+1} [1 + \frac{1}{2}u(t)]^{-1} [1 - \frac{1}{2}(x - 1)u(t)]^{-a - \beta - 1},
\]

where

\[
u(t) = \frac{1}{2}t[t + \sqrt{t^2 + 4)].\]

The formula \( (24) \) appears in Brown's recent paper referred to earlier.

**Added in Proof.** In a private communication to the author, Professor L. Carlitz suggests that following the method of proof of the formula \( (9) \) one can readily obtain its straightforward generalization in the form

\[
\sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} t^n \sum_{k=0}^{n} \frac{(-n)_k c_k}{(1 + a + bn)_k} \frac{x^k}{k!} = \frac{(1 + v)^{a+1}}{1 - bv} \sum_{k=0}^{\infty} \frac{(-ax)^k}{k!},
\]
where the $c_k$ are arbitrary constants and $v$ is defined by (1). It seems worthwhile to remark here that further extensions of (*) form the subject-matter of our discussion in a forthcoming paper.

References


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