## ON THE CHARACTERIZATION OF $\mathcal{H}(B)$ SPACES

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Let $\mathcal{C}(z)$ be the Hilbert space of all formal power series $f(z) = \sum a_n z^n$ in the indeterminate $z$ with complex coefficients such that

$$\|f\|^2 = \sum \|a_n\|^2 < \infty.$$

If $B(w) = \sum B_n w^n$ converges for $|w| < 1$, the formal power series $B(z) = \sum B_n z^n$ represents a function $B(w)$ defined on the unit disk. If the function $B(w)$ is bounded by 1 on the unit disk, the formal product $B(z)f(z)$ belongs to $\mathcal{C}(z)$ whenever $f(z)$ belongs to $\mathcal{C}(z)$ and $\|Bf\| \leqq \|f\|$ [**1**, Theorem 6]. In this case $\mathcal{H}(B)$ is defined to be the set of power series $f(z)$ in $\mathcal{C}(z)$ such that

$$\|f\|_B^2 = \sup\{\|f + Bg\|^2 - \|g\|^2 : g(z) \in \mathcal{C}(z)\} < \infty.$$

The set $\mathcal{H}(B)$ is a Hilbert space with respect to the $B$-norm [**1**, Theorem 7].

The transformation $Q: f(z) \to [f(z) - f(0)]/z$ takes $\mathcal{H}(B)$ into itself and

$$\|Qf\|_B^2 \leqq \|f\|_B^2 - |f(0)|^2 \tag{1}$$

for each $f(z)$ in $\mathcal{H}(B)$. In Theorems 14 and 15 of [**1**] de Branges and Rovnyak characterize the $\mathcal{H}(B)$ spaces for which equality always holds in (1), but they give no characterization of the remaining $\mathcal{H}(B)$ spaces. The purpose of this paper is to characterize these spaces.

It is known that if equality fails in (1) for some $f(z)$ in $\mathcal{H}(B)$, then $B(z)$ is a nonzero element of $\mathcal{H}(B)$ [**1**, Theorem 16]. Furthermore, if $B(z)$ belongs to $\mathcal{H}(B)$, the transformation $M: f(z) \to zf(z)$ takes $\mathcal{H}(B)$ into itself [**1**, p. 43].

THEOREM 1. *Suppose $B(z)$ belongs to $\mathcal{H}(B)$, where $B(z) \neq 0$. Then $Q$ and $M$ are bounded transformations on $\mathcal{H}(B)$ such that*

(i) $\|Qf\|_B^2 \leqq \|f\|_B^2 - |f(0)|^2$ *whenever $f(z)$ belongs to $\mathcal{H}(B)$, and*

(ii) *the range of $T: f(z) \to \langle f, 1\rangle_B + (MM^* - 1)f(z)$ is one-dimensional.*

PROOF. It remains to show that $M$ is bounded and that the range of $T$ is one-dimensional. Since

$$\langle f, Mg\rangle_B = \langle Qf, g\rangle_B + \langle f, B\rangle_B \langle QB, g\rangle_B$$

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whenever $f(z)$ and $g(z)$ belong to $\mathcal{H}(B)$ [1, Theorem 13], $M$ is the adjoint of the bounded transformation $M^*: f \to Qf + \langle f, B\rangle_B QB$. Hence, $M$ is bounded. It follows from the identity $f(0) = \langle f, 1 - \bar{B}(0)B\rangle_B$ (see [1, p. 28]) that $Tf = \langle f, B\rangle_B B$ and, therefore, that $T$ has one-dimensional range.

THEOREM 2. *Let $\mathcal{H}_0$ be a Hilbert space of formal power series such that $M$ and $Q$ are bounded transformations on $\mathcal{H}_0$ which satisfy*

(i) $\|Qf\|_0^2 \leq \|f\|_0^2 - |f(0)|^2$ *whenever $f(z)$ belongs to $\mathcal{H}_0$, and*

(ii) *the range of $T: f(z) \to \langle f, 1\rangle_0 + (MM^* - 1)f(z)$ is one-dimensional. Then $\mathcal{H}_0$ is equal isometrically to some space $\mathcal{H}(B)$, where $B(z)$ is a nonzero element of $\mathcal{H}(B)$.*

PROOF. Let $\mathcal{H}_1$ be the set of power series in $\mathcal{H}_0$ but with a new inner product defined by

(2) $$\langle f, g\rangle_1 = f(0)\bar{g}(0) + \langle Qf, Qg\rangle_0.$$

It is easy to see that $\mathcal{H}_1$ is a Hilbert space. Clearly, $M$ is bounded on the set of constants in $\mathcal{H}_1$. It follows from (2) that $M$ is bounded on the orthogonal complement of the constants in $\mathcal{H}_1$. Therefore, $M$ is bounded on $\mathcal{H}_1$. Let $M^*$ and $M_1^*$ denote the adjoints of $M$ in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, and let $Q^*$ denote the adjoint of $Q$ in $\mathcal{H}_0$. Then

$$\begin{aligned} \langle f, Mg\rangle_1 &= \langle Qf, g\rangle_0 \\ &= \langle Qf, g(0)\rangle_0 + \langle Qf, MQg\rangle_0 \\ &= \langle Qf, 1\rangle_0 \bar{g}(0) + \langle M^*Qf, Qg\rangle_0 \end{aligned}$$

whenever $f(z)$ and $g(z)$ belong to $\mathcal{H}_1$. It follows that

$$M_1^*: f \to \langle Qf, 1\rangle_0 + MM^*Qf$$

and

$$M_1^*M - 1: f \to \langle f, 1\rangle_0 + (MM^* - 1)f.$$

Therefore, $M_1^*M - 1$ has one-dimensional range. Since also $M_1^*M - 1 \geq 0$, there is a nonzero element $e(z)$ of $\mathcal{H}_1$ such that

(3) $$\langle Mf, Mg\rangle_1 - \langle f, g\rangle_1 = \langle f, e\rangle_1 \langle e, g\rangle_1$$

whenever $f(z)$ and $g(z)$ belong to $\mathcal{H}_1$.

Let $B(z) = \|e\|_1 \|Me\|_1^{-1} e(z)$. The identity (3) yields

(4) $$1 - \|B\|_1^2 = \|e\|_1^2 / \|Me\|_1^2.$$

It follows from (i) that $\|f\|_1^2 \geqq \|f\|^2$ for each $f(z)$ in $\mathcal{H}_1$. Therefore, each $f(z)$ in $\mathcal{H}_1$ represents a function $f(w)$ in the unit disk, and for $w$ fixed, $|w| < 1$, the linear functional $f(z) \to f(w)$ on $\mathcal{H}_1$ is continuous. Hence, there is an element $K_w(z)$ of $\mathcal{H}_1$ such that $f(w) = \langle f, K_w \rangle_1$ for every $f(z)$ in $\mathcal{H}_1$. It follows from (3) and (4) that

$$\begin{aligned}
&\langle f, 1 + \bar{w}M[K_w - \bar{B}(w)B]\rangle_1 \\
&\qquad = f(0) + w\langle f - f(0), M[K_w - \bar{B}(w)B]\rangle_1 \\
&\qquad = f(0) + w[\langle Qf, K_w - \bar{B}(w)B\rangle_1 + \langle Qf, e\rangle_1\langle e, K_w - \bar{B}(w)B\rangle_1] \\
&\qquad = f(0) + w[(f(w) - f(0))/w - B(w)\langle Qf, B\rangle_1 \\
&\qquad\qquad + \langle Qf, e\rangle_1(e(w) - B(w)\langle e, B\rangle_1)] \\
&\qquad = f(w) - wB(w)\langle Qf, B\rangle_1[1 - \|Me\|_1^2\|e\|_1^{-2}(1 - \|B\|_1^2)] \\
&\qquad = f(w)
\end{aligned}$$

whenever $f(z)$ is in $\mathcal{H}_1$. Therefore,

$$K_w(z) = 1 + \bar{w}z[K_w(z) - \bar{B}(w)B(z)]$$

and

$$K_w(z) = [1 - \bar{w}\bar{B}(w)zB(z)]/(1 - \bar{w}z).$$

From the inequality

$$0 \leqq \|K_w\|_1^2 = [1 - |wB(w)|^2]/(1 - |w|^2),$$

it follows that $|wB(w)| < 1$ whenever $|w| < 1$ and, therefore, that $|B(w)| < 1$ whenever $|w| < 1$. Hence, the spaces $\mathcal{H}(B)$ and $\mathcal{H}(A)$ exist, where $A(z) = zB(z)$. But the series $K_w(z)$ belongs to $\mathcal{H}(A)$ whenever $|w| < 1$ and $f(w) = \langle f, K_w\rangle_A$ for each $f(z)$ belonging to $\mathcal{H}(A)$ [**1**, p. 28]. Since the set of finite linear combinations of such series is dense in both $\mathcal{H}_1$ and $\mathcal{H}(A)$, and since the two norms coincide on this set, $\mathcal{H}_1$ is equal isometrically to $\mathcal{H}(A)$. Finally, the spaces $\mathcal{H}(B)$ and $\mathcal{H}(A)$ have the same elements and

$$\langle f, g\rangle_A = f(0)\bar{g}(0) + \langle Qf, Qg\rangle_B \tag{5}$$

whenever $f(z)$ and $g(z)$ belong to $\mathcal{H}(B)$ (see the proof to Theorem 16 of [**1**]). Thus, $\mathcal{H}_0$ and $\mathcal{H}(B)$ have the same elements, and, if $f(z)$ belongs to these spaces it follows from (2) and (5) that $\|f\|_0 = \|Mf\|_1 = \|Mf\|_A = \|f\|_B$.

## Reference

**1.** L. de Branges and J. Rovnyak, *Square summable power series*, Holt, New York, 1966.

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