A NOTE ON A THEOREM OF WEYL

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In 1909, H. Weyl [7] proved that for a selfadjoint operator in Hilbert space, perturbation by addition of a selfadjoint compact operator leaves an "essential" part of the spectrum invariant (namely, the limit points of the spectrum and the points of infinite multiplicity). Coburn [1] has given an extension of this theorem to hyponormal operators. Nieto [3] has obtained a generalization of the theorem to a class of operators in a Banach space. The purpose of this note is to give another interpretation of the theorem of Weyl. Hyponormal and symmetrizable operators satisfy this theorem of Weyl-type. However, these results are apparently weaker than those of Coburn and Nieto. The connection with Coburn's theorem is established.

Let T be a bounded linear operator on a Banach space X. Define the nullity, $n(T)$, as the dimension of $N(T)=\{x\colon Tx=0\}$ and the defect, $d(T)$, as the dimension of $X/R(T)$, where $R(T)$ is the range of T. The index of T is given by $k(T)=d(T)-n(T)$. Let $\mathcal{B}(X)$ denote the bounded linear operators on X, $\mathcal{C}(X)$ the compact linear operators. Let $\sigma(T)$ be the spectrum of the operator T. After Coburn [1], we define the Weyl spectrum by

$$\omega(T) = \bigcap_{C\in\mathcal{C}(X)} \sigma(T+C)$$

Schechter [4] has shown that the complement of $\omega(T)$, $C\omega(T) = \{\lambda\colon k(\lambda-T)=0, \text{ and } d(\lambda-T)<\infty \text{ and } n(\lambda-T)<\infty\}$. Let $\alpha(T)$ denote the ascent of T; that is, the smallest nonnegative such that the null spaces $N(T^n)$ are equal if $n\geqq\alpha(T)$. The descent $\delta(T)$ is the smallest integer such that the ranges $R(T^n)$ are equal if $n\geqq\delta(T)$. Finally, if λ_0 is an isolated point of $\sigma(T)$ that is a pole of the resolvent operator $(\lambda-T)^{-1}$, then λ_0 is a pole of the resolvent of finite rank if the spectral projection associated with $\{\lambda_0\}$ has finite dimensional range (see Taylor [5], [6]).

The following gives a general condition for a theorem of Weyl type to hold.

THEOREM. *If the ascent of $\lambda-T$ is finite for a dense set of points λ, then $\omega(T)$ consist of $\sigma(T)$ except for poles of the resolvent of finite rank.*

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PROOF. Since $\omega(T)$ is the intersection of closed sets, then its complement $C\,\omega(T)$ is open. If $\lambda \in C\,\omega(T)$, then $n(\lambda - T) = d(\lambda - T) < \infty$. If $\lambda_0 \in C\,\omega(T) \cap \sigma(T)$, then there is an open connected set S about λ_0 such that $n(\lambda - T) = d(\lambda - T)$ for $\lambda \in S$. For each $\mu \in S$ with finite ascent, we conclude, by Theorem 4.5 of Taylor [5], that $\alpha(\mu - T) = \delta(\mu - T)$. Now, Theorem 9.4 of Taylor [5] implies that μ is an isolated point of $\sigma(T)$. Thus, $S \cap \rho(T) \neq \emptyset$, where $\rho(T)$ is the resolvent set for T. Theorem 9.6 of Taylor [5] implies that $S \cap \sigma(T)$ consists of isolated points of $\sigma(T)$ that correspond to poles of the resolvent, $(\lambda - T)^{-1}$, of finite rank. Thus $\sigma(T) \cap C\,\omega(T)$ consists of isolated points such that each point is a pole of finite rank of the resolvent.

If λ is an isolated point of $\sigma(T)$ that is a pole of the resolvent of finite rank, then $n(\lambda - T) = d(\lambda - T) < \infty$ by Theorem 9.2 of Taylor [5]. Consequently, $\lambda \notin \omega(T)$ by Schechter's characterization of $C\,\omega(T)$.

COROLLARY 1. *If the spectrum of T is a nowhere dense set, then $\omega(T)$ consists of $\sigma(T)$ except for poles of the resolvent of finite rank.*

The result analogous to Coburn's [1] generalization to hyponormal operators of the Weyl theorem depends on the following lemma. An operator is said to be hyponormal if $A^*A \geqq AA^*$.

LEMMA 1. *If A is a hyponormal operator on a Hilbert space, then A has ascent* 0 *or* 1.

PROOF. Let $x \in N(A^2)$. Then

$$\begin{aligned}((A^*A)^2x, x) &= (AA^*\,Ax, Ax)\\ &\leqq (A^*A\,Ax, Ax) = 0,\end{aligned}$$

since $A^*A \geqq AA^*$. Thus $(A^*A^2x, x) = 0$, which implies that

$$(A^*Ax, A^*Ax) = \|A^*Ax\|^2 = 0.$$

Hence $A^*Ax = 0$. But

$$0 = (A^*Ax, x) = (Ax, Ax) = \|Ax\|^2,$$

implies that $x \in N(A)$, or $N(A^2) \subseteq N(A)$. Thus the ascent is 0 or 1.

COROLLARY 2. *If A is hyponormal, then $\omega(A)$ consists of $\sigma(A)$ except for poles of the resolvent of finite rank.*

PROOF. Since A is hyponormal, $\lambda - A$ is hyponormal for all λ. Hence, $\alpha(\lambda - A)$ is finite for all λ. Theorem 1 now gives the desired conclusion.

The connection between Corollary 2 and Coburn's generalization

of the theorem of Weyl is given in Lemma 2. A point $\lambda \in \sigma(T)$ has finite multiplicity if $n(\lambda - T) < \infty$.

LEMMA 2. *Let T be a hyponormal operator. An isolated point of $\sigma(T)$ of finite multiplicity is a pole of the resolvent of finite rank.*

PROOF. In the proof of Theorem 3.1 of Coburn [1], it is shown that if λ is an isolated point of finite multiplicity of a hyponormal operator, then $k(\lambda - T) = 0$. Now Theorem 9.6 of Taylor [5] implies that λ is a pole of the resolvent of finite rank.

Let ϕ be a continuous isomorphism of a Banach space X into a Hilbert space H such that $\phi(X)$ is dense in H. A bounded linear operator T on X is said to be symmetrizable if the induced operator $\hat{T}\phi(x) = \phi(Tx)$ is symmetric with domain $\phi(X)$. The result analogous to Nieto's [2] theorem is immediate after the lemma.

LEMMA 3. *Let T be a linear operator on X. Suppose T is symmetrizable. Then if* $\operatorname{Im}\lambda \neq 0$, *then* $\alpha(\lambda - T) = 0$ *or* 1.

PROOF. Denote the norm in H by $\|\cdot\|_H$. For a symmetric operator S on H, Kato [2], p. 270, shows that for u in the domain of S

$$\|(\lambda - S)u\|_H^2 = \|(\operatorname{Re}\lambda - S)u\|_H^2 + (\operatorname{Im}\lambda)^2\|u\|_H^2.$$

Consequently, if $(\lambda - T)^2 x = 0$, let $u = \phi((\lambda - T)x)$ and $S = \hat{T}$.

Then $(\lambda - S)u = \phi((\lambda - T)^2 x) = 0$. The above equality implies $\operatorname{Im}\lambda \neq 0$, that $\|\phi[(\lambda - T)x]\|_H = 0$, or $(\lambda - T)x = 0$, since ϕ is an isomorphism.

COROLLARY 3. *If T is symmetrizable, then $\omega(T)$ consists of $\sigma(T)$ except for poles of the resolvent of finite rank.*

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