THE REPRESENTATION OF CHAINABLE CONTINUA WITH ONLY TWO BONDING MAPS

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DEFINITION. A set S of continuous functions of $[0, 1]$ into $[0, 1]$ is called a "complete" set of bonding maps if every chainable continuum can be obtained as the inverse limit of an inverse mapping system each of whose coordinate spaces is $[0, 1]$ and each of whose bonding maps is in S.

It is shown in [1] that a dense set is complete. Jolly and Rogers in [3] demonstrate a complete set with only four elements. It follows from the theorem of Jarník and Knichal [2] that there is a complete set with only two elements. Mahavier proves in [4] that every complete set must have at least two elements and therefore *two* is the minimum number.

In this note, we prove the following:

THEOREM. *There exists a continuous function f of $[0, 1]$ into $[0, 1]$ such that $\{f, \frac{1}{2}f\}$ is complete.*

First we will establish a lemma, the proof of which closely follows [2]. Let C denote the set of all continuous functions of $[0, 1]$ into $[0, 1]$. $e(x) = \frac{1}{2}x$, $e^{n+1}(x) = e(e^n(x))$ and $e^0(x) = x$.

LEMMA. *If $\theta_1, \theta_2 \in C$, then there exists $g \in C$ such that $gege^3 = \theta_1$ and $ge^2ge^3 = \theta_2$.*

PROOF. Let g be defined as follows: $g(x) = 3\cdot 2^{-2} + 2x$ for $0 \leqq x \leqq 2^{-3}$, $g(x) = \theta_2(16x-3)$ for $3\cdot 2^{-4} \leqq x \leqq 2^{-2}$, $g(x) = \theta_1(8x-3)$ for $3\cdot 2^{-3} \leqq x \leqq 2^{-1}$, and g is linear in each of the intervals $[2^{-3}, 3\cdot 2^{-4}]$, $[2^{-2}, 3\cdot 2^{-3}]$, and $[2^{-1}, 1]$. It is now easy to verify that $gege^3 = \theta_1$ and $ge^2ge^3 = \theta_2$.

PROOF OF THE THEOREM. It follows from the lemma and a remark made previously that there exists $g \in C$ such that $\{gege^3, ge^2ge^3\}$ is complete. Thus if M is a chainable continuum, then M can be represented in the form $M \cong ge^{n_1}ge^3ge^{n_2}ge^3ge^{n_3}ge^3ge^{n_4}ge^3 \cdots$. The diagram arrows are omitted to save space. Each of the exponents n_i is 1 or 2. Let $f = ege^3g$ and then by regrouping the bonding maps and deleting the first map, we obtain $M \cong e^{n_1-1}fe^{n_2-1}fe^{n_3-1}f \cdots$. Now for each i, $n_i - 1 = 0$ or 1 and so another regrouping yields $M \cong (a_1f)(a_2f)(a^3f) \cdots$ where for each i, $a_i = 1$ or $\frac{1}{2}$. This completes the proof.

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It should be noted that $\{gege^3, ge^2ge^3\}$ is complete as a consequence of the fact that the collection of all finite compositions of these functions is dense in C. The collection of all finite compositions of the functions f and $\frac{1}{2}f$ is not dense in C since the range of f is a proper subinterval of $[0, 1]$.

References

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