ON THE EXISTENCE OF INCOMPRESSIBLE SURFACES
IN CERTAIN 3-MANIFOLDS

WOLFGANG HEIL

If $M$ is the closure of the complement of a regular neighborhood of a nontrivial knot in $S^3$ then there exists a nonsingular torus $T$ embedded in $M$, which is incompressible (i.e. the inclusion $i: T \to M$ induces a monomorphism $i_*: \pi_1(T) \to \pi_1(M)$). If $F$ is any orientable closed incompressible surface embedded in $M$ then $\pi_1(M)$ contains $\pi_1(F)$ as a subgroup. L. Neuwirth [3, Question T] asks whether the converse is true: If $\pi_1(M)$ contains the group $\pi$ of a closed (orientable) surface of genus $g > 1$, does there exist a nonsingular closed surface $F$ of genus $g$ whose fundamental group is injected monomorphically into $\pi_1(M)$ by inclusion? As a partial answer we show that not for every such $\pi \subset \pi_1(M)$ there exists an incompressible $F \subset M$. The question remains open whether $M$ contains incompressible closed surfaces of genus $> 1$. We show that for torus knots $M$ does not contain such surfaces, by showing that $\pi_1(M)$ does not contain subgroups $\pi$.

1. Isotopic surfaces. Let $M$ be a compact 3-manifold (orientable or nonorientable). A “surface $F$ in $M$” always means a 2-sided embedded surface $F$ in $M$ such that $F \cap \partial M = \partial F$. $F$ is incompressible in $M$ iff $F \neq S^2$ and $\ker(i_*: \pi_1(F) \to \pi_1(M)) = 1$, where $i: F \to M$ is the inclusion. We say $M$ is $P^2$-irreducible iff $M$ is irreducible (every 2-sphere bounds a ball) and does not contain (2-sided) projective planes. $M$ is called boundary-irreducible iff $\partial M$ is a system of incompressible surfaces.

Theorem 1. Let $M$ be a $P^2$-irreducible 3-manifold. Let $G$ be an incompressible surface in $M$ and $\pi \subset i_*\pi_1(G) \subset \pi_1(M)$. If there exists an incompressible surface $F \subset M$ such that $\partial F \subset \partial G \cap \partial F$ and $i_*\pi_1(F) = \pi$, then $F$ is isotopic to $G$.

This follows from theorems obtained by Waldhausen [5]. In particular we need the following:

Proposition [5, Proposition 5.4]. Let $M$ be $P^2$-irreducible. Let $F$ and $G$ be incompressible surfaces in $M$, $\partial F \subset \partial F \cap \partial G$, such that $F \cap G$ consists of mutually disjoint simple closed curves (with transversal intersection at any curve which is not in $\partial F$). Let $H$ be a surface and suppose there is a map $f: H \times I \to M$ such that $f| H \times 0$ is a covering map onto $F$.

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and \( f(\partial (H \times I) - H \times 0) \subset G \). Then there is a surface \( \tilde{H} \) and an embedding \( \tilde{H} \times I \to M \) such that \( \tilde{H} \times 0 = \tilde{F} \subset F \); \( \text{Cl}(\partial (\tilde{H} \times I) - \tilde{H} \times 0) = \tilde{G} \subset G \) and \( \tilde{F} \cap \tilde{G} = \partial \tilde{F} \); moreover if \( \tilde{G} \cap \tilde{F} \neq \partial \tilde{G} \) then \( \tilde{F} \) and \( \tilde{G} \) are discs.

Waldhausen proves this for orientable \( M, F, G \), using his Lemmas 5.1 to 5.3 in [5]. In the nonorientable case 5.1 of [5] may be proved by looking at the orientable 2-sheeted covering of \( M \) (see [2]). Then the proofs of Lemmas 5.2 to 5.4 in [5] go through in the nonorientable case as well, noting that \( F \) and \( G \) are 2-sided in \( M \).

**Proof of the theorem.** Suppose \( F \) exists. By small isotopic deformations, constant on \( \partial M \), we may assume that \( G \cap F \) consists of a system of closed curves, the number of which is minimal. We claim:

There exists a surface \( H \) homeomorphic to \( F \) and a map \( f : H \times I \to M \) such that \( f|_H \times 0 \) is a homeomorphism onto \( F \) and \( f|_H \times 1 \cup \partial H \times I \subset G \). For, let \( f|_H \times 0 = i : F \to M \). Since \( 0 \leq i_\ast(G) \) and \( \partial F \subset \partial F \cap \partial G \), we can define the map on \( H \times 0 \cup \partial H \times I \cup H^{(1)} \times I \), where \( H^{(1)} \) is the 1-skeleton of \( H \), such that \( f|_H \times 1 \subset G \). Since \( G \) is incompressible, we can extend this map to a map from \( \partial (H \times I) \to M \). Now \( \pi_2(M) = 0 \) (by our assumption on \( M \) and the projective plane theorem [1]; in fact it follows from the Hurewicz-isomorphism on the universal cover that \( M \) is aspherical), therefore \( f \) can be extended to a map \( H \times I \to M \). The rest of the proof copies the proof of Corollary (5.5) in [5]: by the proposition, there exist pieces \( \tilde{G} \subset G \) and \( \tilde{F} \subset F \) which are parallel in \( M \) such that \( \tilde{F} \cap \tilde{G} = \partial \tilde{F} \). If \( \tilde{G} \cap \tilde{F} \neq \partial \tilde{G} \) then \( \tilde{F} \cup \tilde{G} \) bounds a ball, since \( M \) is irreducible. This ball contains a piece \( F' \subset F \). Deforming \( F' \) out of this ball across \( \tilde{G} \), we could make \( \tilde{F} \cap \tilde{G} \) smaller, a contradiction. Hence we have \( \tilde{G} \cap \tilde{F} = \partial \tilde{G} \). Therefore there exists an isotopic deformation of \( F \) (constant on \( F \setminus \tilde{F} \)) which throws \( \tilde{F} \) onto \( \tilde{G} \). If \( \tilde{F} \) would not be all of \( F \), then we could deform \( \tilde{F} \setminus \partial F \cap \tilde{G} \) out of \( G \) (keeping \( \partial F \) fixed) and thereby reduce the intersection number \( F \cap G \). Hence \( F = \tilde{F} \), \( F \cap G = \partial F \subset \partial F \cap \partial G \), hence \( \partial \tilde{G} = \tilde{G} \cap F \subset \partial M \) and since \( G \cap \partial M = \partial G \) we have \( G = \tilde{G} \).

Let \( \mathfrak{F} \) be a subgroup of \( \pi_1(M) \). We say \( \mathfrak{F} \) is carried by a surface \( F \subset M \) iff there exists an embedding \( i : F \to M \) such that \( i_\ast \pi_1(F) = \mathfrak{F} \) and \( \ker i_\ast = 1 \).

**Corollary.** Let \( M \) be \( P^2 \)-irreducible. Let \( G \) be a closed incompressible surface of genus \( > 1 \) in \( M \). Then there exists a subgroup \( \mathfrak{F} \subset \pi_1(M) \) which is not carried by a surface \( F \subset M \) but is isomorphic to \( \pi_1(F) \). (In fact, if \( G \) is not a Klein bottle there exist infinitely many non-isomorphic subgroups of \( \pi_1(M) \) having this property.)

**Proof.** Let \( F \) be a finite covering of \( G \) such that \( F \) is not homeomorphic to \( G \). (Since \( G \neq S^2, P^2, \text{Torus}, \text{Klein bottle} \), we can construct
infinitely many topologically different compact $F$'s.) Then $p_*\pi_1(F) = \mathfrak{F}$ (where $p: F \to G$ is the covering map) is a subgroup of $\pi_1(G)$, hence of $i_*\pi_1(G) \subset \pi_1(M)$. If $\mathfrak{F}$ would be carried by $F$, then by Theorem 1, $F$ would be isotopic to $G$, a contradiction.

In particular this corollary applies to complements of nontrivial knots as mentioned in the introduction.

2. **Surfaces in 3-manifolds which groups have a center.** Let $\mathfrak{F}$ be the fundamental group of a closed surface $F$. If $F$ is orientable suppose genus $(F) > 1$, if $F$ is nonorientable let genus $(F) > 2$.

**Lemma.** Let $M$ be an irreducible (compact) 3-manifold with $\pi_1(M) \approx \mathfrak{F} \times \mathbb{Z}$, then $M$ is a fibre bundle over $S^1$ with fiber $F$.

This is a special case of Stallings theorem [4].

**Theorem 2.** Let $M$ be a $P^2$-irreducible, boundary irreducible 3-manifold and suppose the center $\mathfrak{F}$ of $\pi_1(M)$ is infinite. If $\partial M \neq \emptyset$, then $\pi_1(M)$ does not contain a subgroup $\mathfrak{F}$ as above.

**Proof.** Suppose there exists $\mathfrak{F} \subset \pi_1(M)$. Then, since the center of $\mathfrak{F}$ is trivial, $\mathfrak{F} \cap \mathfrak{F} = 1$. If $t \in \mathfrak{F}$ is of infinite order, the subgroup in $\pi_1(M)$ which is generated by $\mathfrak{F}$ and $t$ is isomorphic to $\mathfrak{F} \times \mathbb{Z}(t)$. If $D(M)$ denotes the double of $M$, then since $M$ is boundary irreducible, $i_*: \pi_1(M) \to \pi_1(D(M))$ is a monomorphism, where $i: M \to D(M)$ is the inclusion (this is well known; a proof may be found, e.g., in [4]). Since $D(M)$ is $P^2$-irreducible and $\pi_1(D(M))$ not finite, $D(M)$ is aspherical (see the remark in the proof of Theorem 1). Therefore we can construct a map $f: F \times S^1 \to D(M)$ which induces the embedding $\mathfrak{F} \times \mathbb{Z} \to \pi_1(M) \to \pi_1(D(M))$. It follows from Waldhausen's theorem [5, Theorem 6.1] (see [2] for the nonorientable case), that $f$ is homotopic to a covering map. In particular, since $F \times S^1$ is compact it follows that $\mathfrak{F} \times \mathbb{Z}$ has finite index in $\pi_1(D(M))$ and therefore in $\pi_1(M) \subset \pi_1(D(M))$. Now consider the covering $\tilde{M}$ of $M$ which is associated to $\mathfrak{F} \times \mathbb{Z}$. $\tilde{M}$ is compact. Now the universal covering of $M$ can be embedded in a ball such that the interior of this ball is contained in the embedding ([5, Theorem 8.1]; the proof in the non-orientable case is quite similar, since the only thing needed is the existence of a hierarchy [2]). Hence $\tilde{M}$ does not contain fake 3-cells, and since $\pi_2(\tilde{M}) = 0$ it follows that $\tilde{M}$ is irreducible. By the lemma, $\tilde{M}$ is a fiber bundle with fiber $F$, in particular $\tilde{M}$ is closed, which is absurd.

The first part of the proof gives us immediately:

**Proposition.** Let $M$ be a closed $P^2$-irreducible 3-manifold and sup-
pose the center $\mathfrak{Z}$ of $\pi_1(M)$ is infinite. If $\pi_1(M)$ contains a subgroup $\mathfrak{Y}$ then $F \times S^1$ is a covering of $M$.

**Corollary (to Theorem 2).** The groups

$$
| t_i, \ldots, t_m, g_1, \ldots, g_n, a_1, b_1, \ldots, a_p, b_p, h:
$$

$$
l_i h t_i^{-1} = h; g_i h g_i^{-1} = h; a_i h a_i^{-1} = h; b_i h b_i^{-1} = h;
$$

$$
g_i^\alpha h^\beta = 1, (\alpha_i, \beta_i) = 1, t_1 \cdots t_m g_1 \cdots g_n \prod_{i=1}^{p} [a_i, b_i] = h^b, \ b \in \mathbb{Z}|
$$

do not contain a subgroup $\mathfrak{Y}$.

These are fundamental groups of Seifert fiber spaces. In particular the groups of torus knots $| g, h: g^a h^b = 1 |$ do not contain a subgroup $\mathfrak{Y}$. Hence the complement of a torus knot does not contain closed incompressible surfaces other than Tori.

**Remark.** The nonexistence of closed surfaces of genus $>1$ in irreducible orientable 3-manifolds $M$ with nonempty boundary for which $\pi_1(M)$ has nontrivial center follows immediately from Waldhausen’s papers [6], [7]. In [6] Waldhausen proves that these manifolds are Seifert fiber spaces and in [7,§(10.3)] it is remarked that any incompressible surface in $M$ which is not boundary-parallel either consists of Seifert fibers (but does not contain singular fibers) or is a branched covering over the Seifert surface (“Zerlegungsfläche”).

**References**


Rice University