ON A CONJECTURE OF CHABAUTY

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Introduction. Let $G$ be a connected Lie group. By a lattice of $G$, we mean a discrete subgroup $\Gamma$ of $G$ such that $G/\Gamma$ has a finite invariant measure. The set of all lattices of $G$ is denoted by $\mathcal{L}(G)$. In [1], Chabauty introduced the notion of limit of subgroups of $G$. A sequence $\{H_n\}$ of subgroups of $G$ converges to a subgroup $H$ if for any given compact subset $K$ of $G$ and neighborhood $V$ of the identity $e$ in $G$, $H \cap K \subseteq VH_n$ and $H_n \cap K \subseteq VH$ hold for sufficiently large $n$. Thus $\mathcal{L}(G)$ becomes a topological space with the Chabauty topology defined by limit of lattices. In [5], some topological properties of $\mathcal{L}(G)$ have been studied. $\mathcal{L}(G)$ is separable metric. However in general we do not know whether $\mathcal{L}(G)$ is locally compact or not. Let $A(G)$ be the group of all continuous automorphisms of $G$. Equipped with the compact-open topology, $A(G)$ is a Lie group. $A(G)$ operates continuously on $\mathcal{L}(G)$ with operation defined by $(\alpha, \Gamma) \mapsto \alpha(\Gamma)$, for $\alpha \in A(G)$ and $\Gamma \in \mathcal{L}(G)$. In [1], Chabauty conjectured that for any lattice $\Gamma$ of $G$, $A(G)\Gamma$ with induced topology from $\mathcal{L}(G)$ is homeomorphic to $A(G)/N(\Gamma)$, where $N(\Gamma)$ is the isotropy subgroup at $\Gamma$, or equivalently $A(G)\Gamma$ is locally compact. Followed by a theorem of Malcev [3], the conjecture is true for nilpotent Lie groups. For semisimple Lie groups, the author obtained some partial results in [5]. The purpose of this paper is to construct a counterexample in the case of solvable Lie groups.

1. Semidirect product of a compact group and a vector group. Let $V = \mathbb{R}^n$ and $K$ a compact subgroup of $\text{GL}(n, \mathbb{R})$. In $G = K \times V$ (in the sense of set only), we define a group structure by

$$(k, v)(k_1, v_1) = (kk_1, k_1^{-1}v + v_1)$$

for $k, k_1 \in K$ and $v, v_1 \in V$.

Lemma 2. $V$ is the nilpotent radical of $G$, i.e., the maximal connected normal nilpotent subgroup of $G$.

Proof. Let $n(G)$ be the nilpotent radical of $G$. Clearly $n(G) \supset V$. Hence $n(G) = (n(G) \cap K) \cdot V$. Since $n(G) \cap K$ is compact, it is central in $n(G)$. However the action of $K$ on $V$ is faithful. It implies that $n(G) \cap K = \{e\}$. Thus $V = n(G)$.

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As $V$ is a characteristic subgroup of $G$, we have then the restriction map $\text{res}: A(G) \rightarrow \text{GL}(V)$ and the induced map $\text{ind}: A(G) \rightarrow A(K)$. Let $C(K)$ be the centralizer of $K$ in $\text{GL}(n, \mathbb{R})$ and $A(G)^0$ the identity component of $A(G)$.

**Lemma 3.** If $K$ is commutative, then $\text{res}(A(G)^0) = C(K)^0$.

**Proof.** Since $K$ is compact and abelian, it is well known $A(K)^0 = \{e\}$. Hence $\text{ind}(A(G)^0) = \{e\}$. Clearly

$$(e, (\alpha k)x) = \alpha((e, kx)) = \alpha((k, e)(e, x)(k, e)^{-1})$$

$$= (k, e)\alpha((e, x))(k, e)^{-1} = (e, (k\alpha)x)$$

for all $\alpha \in A(G)^0$, $k \in K$, $x \in V$. Therefore $\text{res}(A(G)^0) \subseteq C(K)^0$. Conversely given any $\beta \in C(K)$, we define $\overline{\beta}((k, x)) = (k, \beta x)$, $(k, x) \in G$. It is obvious that $\overline{\beta} \in A(G)$ and $\text{res}(\overline{\beta}) = \beta$. Thus $\text{res}(A(G)^0) = C(K)^0$.

4. **Diophantine approximation.** Let $\gamma$ be an irrational number. As an immediate consequence of diophantine approximation, there exists an increasing sequence $(a_n)$ of positive integers such that $\langle a_n \gamma - \lfloor a_n \gamma \rfloor \rangle$ converges to zero, where $\lfloor x \rfloor$ is the function of the greatest integer $\leq x$, $x \in \mathbb{R}$. Denote $\lfloor a_n \gamma \rfloor$ and $a_n \gamma - \lfloor a_n \gamma \rfloor$ by $b_n$ and $c_n$ respectively. By an easy computation, we have

$$\begin{pmatrix}
1 & 0 & 0 & 0
0 & 1 & 0 & 1 + a_n - a_n
0 & 0 & 1 & 0
0 & 0 & -1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & \gamma & 0
0 & 1 & 0 & 0
0 & 0 & 1 & a_n
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \gamma & c_n
0 & 1 & 0 & 0
0 & 0 & 1 & 0
0 & 0 & 0 & 1
\end{pmatrix}.$$
Let
\[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & 0 & X \\
0 & 0 & X \\
\end{pmatrix}
\]
\[\lambda \in R - \{0\}\text{ and } X \in GL(2, R).\]

Due to our construction, \(\tilde{\gamma}_n \rightarrow \tilde{\gamma}\) and \(\tilde{\gamma}_n \neq \tilde{\gamma}_m\) for \(n \neq m\). It is very easy to see that
\[
C(Q)\tilde{\gamma}_n \neq C(Q)\tilde{\gamma}_m \quad \text{for } n \neq m.
\]

**Proposition 5.** \(C(Q)\tilde{\gamma} \text{SL}(4, R) = \{x\tilde{\gamma}y \mid x \in C(Q)\tilde{\gamma} \text{ and } y \in \text{SL}(4, R)\}\) is not locally closed.

**Proof.** Suppose false. Then \(C(Q)\tilde{\gamma}\) is open in \(C(Q)\tilde{\gamma} \text{SL}(4, R)\) and \(\tilde{\gamma}_n \rightarrow \tilde{\gamma}\). This implies \(C(Q)\tilde{\gamma}_n = C(Q)\tilde{\gamma}\) for sufficiently large \(n\). However this contradicts the fact that \(C(Q)\tilde{\gamma}_n \neq C(Q)\tilde{\gamma}_m\) for \(n \neq m\).

6. A counterexample. Let \(V = R^4, K = \tilde{\gamma}^{-1}Q\tilde{\gamma}, G = K \cdot V\) as constructed in §1, and \(\Gamma = Z^4 \subset R^4\). Since \(G/\Gamma\) is compact, certainly \(\Gamma\) is a lattice of \(G\).

**Main Theorem.** \(A(G)\Gamma\) is not locally compact with induced topology \(\mathcal{S}(G)\).

**Proof.** Suppose false. Then \(A(G)^0\Gamma\), open in \(A(G)\Gamma\) is locally compact. By Lemma 3, \(A(G)^0\Gamma = \text{res}(A(G)^0)\Gamma = C(K)^0\Gamma\). Let \(S_\ast(G)\) be the set of all lattices of \(G\) contained in \(V\). It is clear that \(S_\ast(G)\) is a closed subset of \(S(G)\) and \(S_\ast(G) = S(V)\). It is a well-known classical result \(S(V) \approx GL(V)^0/N(\Gamma)\), where \(N(\Gamma) = \text{SL}(4, Z)\). Therefore it follows that \(C(K)^0 \text{SL}(4, Z)\) is locally compact. But \(C(K)^0 \text{SL}(4, Z) = \tilde{\gamma}^{-1}(C(Q)\tilde{\gamma} \text{SL}(4, Z))\) is not locally compact by Proposition 5. Thus we are led to contradiction.

A Remark. Although Chabauty's conjecture is not true in solvable Lie groups, it is still very likely that the conjecture will be valid in semisimple Lie groups supported by some indications in [5]. If \(G\) is semisimple with each factor of \(R\)-rank \(\geq 2\), \(S(G)\) is locally compact which is an easy consequence of [7].
Reference


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