

## A NEW CRITERION FOR UNIFORM DISTRIBUTION

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**1. A criterion for uniform distribution.** Let  $[a, b]$  be an interval which is a subset of the interval  $[0, 1]$ . We denote this interval by  $P$  and say it has length  $p = b - a$ . Let  $\chi_1, \chi_2, \dots, \chi_k, \dots$  be a finite or infinite sequence of points in the interval  $[0, 1]$ . For any integer  $n > 0$ , we say that the ratio measure of  $P$ ,  $R_n(P)$ , is the number of points in the finite partial sequence  $\chi_1, \chi_2, \dots, \chi_n$  which lie in the interval  $P$  divided by the number  $n$  of points in the sequence.

DEFINITION. The infinite sequence  $\chi_1, \chi_2, \dots, \chi_k, \dots$  is said to be "uniformly distributed" if for any  $p$ -length interval  $P$ , a subset of  $[0, 1]$ ,

$$\lim_{n \rightarrow \infty} R_n(P) = p.$$

A necessary and sufficient condition that a sequence be uniformly distributed already exists [1]. This paper offers a new, elementary and perhaps more intuitive condition.

DEFINITION. Let there be given a set of points  $\{a_1, a_2, \dots, a_n\}$  in the interval  $[0, 1]$ ,  $a_1 < a_2 < \dots < a_n$ . The set is said to be an "almost-arithmetic progression- $(\delta, \epsilon)$ " if the following properties hold: There exists an  $\eta \leq \epsilon$  such that

(1.1)  $a_1$  lies in the interval  $[0, \eta + \delta\eta]$ ,

(1.2) for  $k = 1, 2, \dots, n - 1$ , the point  $a_{k+1}$  lies in the interval  $[a_k + \eta - \delta\eta, a_k + \eta + \delta\eta]$ ,

(1.3)  $a_n$  lies in the interval  $[1 - \eta - \delta\eta, 1]$ .

An almost-arithmetic progression- $(\delta, \epsilon)$  will be referred to as an almost-arithmetic progression when there can be no confusion, and may be abbreviated as an "a.-a.p."

The motivation for the definition of an almost-arithmetic progression- $(\delta, \epsilon)$  is in the following lemma.

LEMMA 1.1. *Given any  $\epsilon''$ , there exist  $\delta, \epsilon$  such that for any interval  $P$  of length  $p$  and any almost-arithmetic progression- $(\delta, \epsilon)$ ,  $a_1, a_2, \dots, a_n$ , the ratio measure  $R_n(P)$  falls within  $\epsilon''$  of  $p$ .*

PROOF. The proof is elementary and will be omitted.

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**THEOREM 1.2.** *The infinite sequence of points  $\chi_1, \chi_2, \dots, \chi_n, \dots$ , in  $[0, 1]$ , is uniformly distributed iff the following condition holds: Given  $\delta, \epsilon$ , and  $\epsilon'$  all greater than zero, there exists an  $N$  such that for  $n > N$ , the partial sequence  $\chi_1, \chi_2, \dots, \chi_n$  may be partitioned into a set of disjoint almost-arithmetic progressions- $(\delta, \epsilon)$  with at most  $n_0$  points left over,  $n_0 < \epsilon'n$ .*

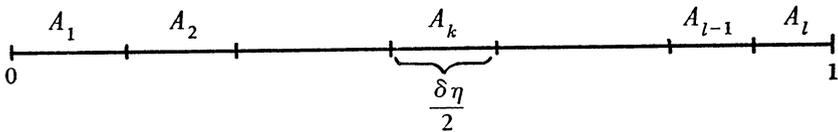
**PROOF OF SUFFICIENCY.** Given  $\epsilon''$ , choose  $\delta$  and  $\epsilon$  as in Lemma 1.1. Take any  $\epsilon'$  and find  $N$ . If  $n > N$ , we know that  $n - n_0$  points of the sequence  $\chi_1, \chi_2, \dots, \chi_n$  are partitioned into almost-arithmetic progressions- $(\delta, \epsilon)$ . For any interval  $P$  in  $[0, 1]$ , each of these a.-a.p's gives a ratio measure  $R_v(P)$  which falls within  $\epsilon''$  of  $p$  by Lemma 1.1,  $v$  being the number of points in the a.-a.p. If we consider the ratio measure  $R_{n-n_0}(P)$  using all points which occur in a.-a.p's, the quantity will be a convex average of each single ratio measure of a particular a.-a.p. and will therefore fall within  $\epsilon''$  of  $p$ . The remaining  $n_0$  points can only serve to increase the difference of the ratio measure,  $R_n(P)$ , from  $p$  by the quantity  $\epsilon'$ .

Therefore we see

$$(1.4) \quad |R_n(P) - p| < \epsilon' + \epsilon''$$

and since the right-hand side of (1.4) may be made as small as we wish for  $N$  sufficiently large, the sequence is uniformly distributed.

**PROOF OF NECESSITY.** Let  $\chi_1, \chi_2, \dots, \chi_k, \dots$  be an infinite sequence, uniformly distributed on  $[0, 1]$ . We will attempt to show that the proposed necessary condition holds. Choose  $\delta, \epsilon$ , and  $\epsilon'$ , where we take  $\delta$  to be  $1/m$  for some integer  $m$  without loss of generality. Now divide the interval  $[0, 1]$  into intervals  $A_1, A_2, \dots, A_l$ , each of length  $(\delta\eta)/2$  which together cover the interval  $[0, 1]$  and overlap only at endpoints.



We have chosen  $\eta < \epsilon$  so that there will be an integral number of such intervals  $A_i$ . That is  $2/\delta \cdot \eta = l$ , an integer.

Now since  $\{\chi_k\}$  is uniformly distributed, given  $\epsilon_0$ , for all  $i$ , we may choose an  $N_i$ , so that if  $n \geq N_i$  we always have

$$(1.5) \quad \delta\eta/2 - \epsilon_0 \leq R_n(A_i) \leq \delta\eta/2 + \epsilon_0.$$

Let  $N = \max_{i=1, \dots, l} N_i$ . We now have that for  $n \geq N$ , equation (1.5) holds for all  $i = 1, \dots, l$ .



$$\begin{array}{cccc}
 \chi_1 & \chi_{l+1} & \cdots & \chi_{m_1 l+1} \\
 \chi_2 & \chi_{l+2} & \cdots & \chi_{m_2 l+2} \\
 & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot \\
 \chi_l & \chi_{2l} & \cdots & \chi_{m_l \cdot l}
 \end{array}$$

Looking at each row in the above array, it is clear that the first element may fall somewhere in the interval  $[0, 1]$ , the second a distance  $\eta \leq 1/m < \epsilon$  to the right until an element comes within  $\eta$  of 1. The elements so far have not gone into an a.-a.p. But the next element will fall within  $\eta$  of 0; successive elements will move to the right in steps of  $\eta$  until the last element is within  $\eta$  of 1. The sequence which started at zero is an almost-arithmetic progression- $(\delta, \epsilon)$ . (In fact,  $\delta$  is zero; it is a true arithmetic progression.) The next element of the sequence will be the first of a new a.-a.p. This process will continue until the sequence ends, perhaps stopping at a distance greater than  $\eta$  from 1. Now those elements which did not fall in an a.-a.p. may be bounded above in number, with the bound depending only on  $a$  and the integer  $m$ . Multiplying by  $l$ , we will have an upper bound  $B$  not depending on  $n$  (the number of points in the partial sequence) for the number of points which do not fall in a.-a.p.'s, i.e.,  $n_0 \leq B$ . We may increase  $n$  until  $B < \epsilon'n$  and we are finished.

It is obvious that only crude approximations were used here, and it would seem that good lower bounds for the actual speed of convergence of the ratio measure  $R_n$  to the measure of an arbitrary interval  $P$  may be achieved for specific irrationals  $a$  by finer applications of this method.

EXAMPLE 2 (FROM PÓLYA AND SZEGÖ [2]).  $\chi_k = ak^\sigma - [ak^\sigma]$ ,  $a > 0$ ,  $0 < \sigma < 1$  is uniformly distributed. It is left as an easy exercise that if  $\delta, \epsilon$  are given, we may choose  $N'$  such that for  $n \geq N'$ , then for all  $k$ ,  $n \leq ak^\sigma \leq n+1$ , the set of points  $\chi_k$  thus chosen form an almost-arithmetic progression- $(\delta, \epsilon)$ . All the points  $\chi_k$  such that  $ak^\sigma < N'$  may be thrown away, together with all the points which overlap at the end so long as  $N - N' \gg N'$  ( $N$  is the  $N$  of the criterion).

EXAMPLE 3 (FROM PÓLYA AND SZEGÖ).  $\chi_n = a(\log n)^\sigma - [a(\log n)^\sigma]$ ,  $a > 0$ ,  $\sigma > 1$ . We may use the same method as Example 2. After some experience with the criterion, the check of uniform distribution becomes a simple mental calculation for sequences of this type. Note that if  $\sigma = 1$  in the above example, uniform distribution does not hold. The intuition supplied by this criterion immediately suggests a proof of the fact.

## BIBLIOGRAPHY

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