A NEW PROOF OF THE TRANSPPOSITION THEOREM

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It is well known that the minimax theorem can be deduced from various forms of the transposition theorem (see e.g. [2] and [3]). In particular it follows from the following:

**Transposition Theorem.** Let $A$ be a real $m$ by $n$ matrix and $A'$ its transpose; let $x$ and $y$ denote respectively $n$- and $m$-dimensional real column vectors. Then either $Ax \geq 0$ for some $x \neq 0$, or $A'y \leq 0$ for some $y \geq 0$, $y \neq 0$ ($x = (x_1, \ldots, x_n) \geq 0$ means $x_i \geq 0$, $0 \leq i \leq n$).

It is shown in [3] that the above theorem follows readily from

**Stiemke's Theorem** [4]. If $S$ is a subspace of $R^n$ and $S^+$ the orthogonal complement of, then $S \cap S^+$ contains some vector $x \geq 0$, $x \neq 0$.

In this note we obtain a formula for the number of orthants intersected by a subspace of $R^n$. Stiemke's theorem and ipso the above mentioned transposition theorem will be obtained as a direct consequence of the formula. We employ the following terminology. The hyperplanes $H_1, \ldots, H_s$ of $R^n$ ($s \geq n$) are said to be in general position if the intersection of any $n$ of them is 0. The $k$-dimensional subspace $S$ of $R^n$ is said to be in general position if the $n$ subspaces $H_i \cap S$, where $H_i = \{x | x_i = 0\}$ ($1 \leq i \leq n$), are hyperplanes of $S$ in general position. Letting $H_{i_1} \cap \cdots \cap H_{i_k} = 0$ or equivalently, $R^n = S \oplus H_{i_1} \oplus \cdots \oplus H_{i_k}$ for all choices of $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If the $k$ vectors $s_i = (s_{i1}, \ldots, s_{in})$ ($1 \leq i \leq k$) form a basis for $S$, then it is easily checked that $S$ is in general position if and only if all $k$ by $k$ minors of $(s_{ij})$ ($1 \leq i \leq k$, $1 \leq j \leq n$) are $\neq 0$. An orthant of $R^n$ is defined to be the set $\{x | x_1 > 0, \ldots, x_n > 0\}$ where $\{e_i\}$ denotes any fixed choice of $1$'s. We now prove the following:

**Theorem.** Let $S$ be a $k$-dimensional subspace of $R^n$ which is in general position. The number of orthants intersected by $S$ is

$$2 \sum_{i=0}^{k-1} \binom{n-1}{i}.$$
PROOF. For \( k = 1 \) and \( n = k \), the result is obvious so that we may assume \( 2 \leq k \leq n \). Let \( 0_{n,k} \) = number of orthants intersected by \( S \) (\( 2 \leq k < n \)). Let \( R_{n,k} \) = number of regions into which \( n \) hyperplanes of \( R^k \) in general position decompose \( R^k \) (\( n \geq 1, k \geq 2 \)). The orthants of \( R^n \) intersected by \( S \) stand in 1-1 correspondence with the regions into which the hyperplanes \( H_i \cap S \) (\( 1 \leq i \leq n \)) decompose \( S \). Identifying \( S \) with \( R^k \) we obtain \( 0_{n,k} = R_{n,k} \) (\( 2 \leq k < n \)). The formula for \( R_{n,k} \) is well known and can be found in [1]. We give another derivation of this formula.

Let \( L_1, \ldots, L_{n+1} \) be \( n+1 \) hyperplanes of \( R^k \) in general position decomposing \( R^k \) (\( k \geq 3 \)) into \( R_{n+1,k} \) \( k \)-dimensional regions. Let \( L_{n+1} \) intersect \( c \) of these. Each of the regions is divided by \( L_{n+1} \) into two regions so that \( R_{n+1,k} = R_{n,k} + c \). Let \( L_i' = L_i \cap L_{n+1} \) (\( 1 \leq i \leq n \)). Then \( L_1', \ldots, L_n' \) are hyperplanes of \( L_{n+1} \) in general position decomposing \( L_{n+1} \) into \( R_{n,k-1} \) (\( k - 1 \))-dimensional regions. These regions are in 1-1 correspondence with the \( c \) \( k \)-dimensional ones intersected by \( L_{n+1} \). Thus:

\[
(1) \quad R_{n+1,k} = R_{n,k} + R_{n,k-1} \quad (k \geq 3, n \geq 1).
\]

Let \( R_k(z) = \sum_{n=1}^{\infty} R_{n,k} z^n \). Since \( R_1,k = 2 \) we conclude from (1) that

\[
(2) \quad R_k(z) = R_{k-1}(z)(z/(1 - z)) + 2z/(1 - z) \quad (k \geq 3).
\]

Since \( R_n,2 = 2n \) we have \( R_2(2) = 2z/(1 - z)^2 = 2[z/(1 - z) + z^2/(1 - z)^2] \). Repeated use of (2) yields

\[
(3) \quad R_k(z) = 2[z/(1 - z) + \cdots + z^k/(1 - z)^k],
\]

Equating the \( n \)th coefficients of both sides of (3), we obtain the desired formula

\[
R_{n,k} = 2 \sum_{i=0}^{k-1} \binom{n-1}{i}.
\]

We now obtain Stiemke's theorem as a consequence of the above formula. We may assume that \( S \) is in general position; the case where \( S \) is not in general position is then treated by a standard limiting argument.

Let \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_{n-k} \leq n \) denote \( n \) integers comprising all integers from 1 to \( n \). \( S \oplus H_{i_1} \cdots H_{i_k} = R^n \) as \( S \) is in general position. Hence \( 0 = (S \oplus H_{i_1} \cdots H_{i_k}) \cap H_{j_1} \cdots H_{j_{n-k}} = S \cap H_{j_1} \cdots H_{j_{n-k}} \). Since \( S \cap H_{j_1} \cdots H_{j_{n-k}} = 0 \) for all choices of \( 1 \leq j < \cdots < j_{n-k} \leq n \), we conclude that \( S \cap \) is in general position. Thus the number of orthants intersected by \( S \cap \) is \( 2 \sum_{i=0}^{n-1-k} \binom{n-1}{i} \). If \( x, y \) are in the same orthant then \( x \cdot y > 0 \) so that the orthants intersected by \( S \) are distinct from those.
intersected by $S^\perp$. Since there are $2^n$ orthants and

$$2 \sum_{i=0}^{k} \binom{n-1}{i} + 2 \sum_{i=k}^{n-1} \binom{n-1}{i} = 2 \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^n,$$

we conclude that every orthant is intersected by $S$ or $S^\perp$. In particular this holds for the positive orthant, thus proving Stiemke's theorem.

**References**