A NEW PROOF OF THE TRANSPOSITION THEOREM

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It is well known that the minimax theorem can be deduced from various forms of the transposition theorem (see e.g. [2] and [3]). In particular it follows from the following:

TRANSPOSITION THEOREM. *Let A be a real m by n matrix and A' its transpose; let x and y denote respectively n- and m-dimensional real column vectors. Then either $Ax \geqq 0$ for some $x \geqq 0$, $x \neq 0$ or $A'y \leqq 0$ for some $y \geqq 0$, $y \neq 0$ ($x = (x_1, \cdots, x_n) \geqq 0$ means $x_i \geqq 0$, $0 \leqq i \leqq n$).*

It is shown in [3] that the above theorem follows readily from

STIEMKE'S THEOREM [4]. *If S is a subspace of R^n and S^+ the orthogonal complement of, then $S \cup S^+$ contains some vector $x \geqq 0$, $x \neq 0$.*

In this note we obtain a formula for the number of orthants intersected by a subspace of R^n. Stiemke's theorem and ipso the above mentioned transposition theorem will be obtained as a direct consequence of the formula. We employ the following terminology. The hyperplanes $H_1, \cdots, H_s$ of R^n ($s \geqq n$) are said to be in general position if the intersection of any n of them is 0. The k-dimensional subspace S of R^n is said to be in general position if the n subspaces $H_i \cap S$, where $H_i = \{x \mid x_i = 0\}$ $(1 \leqq i \leqq n)$, are hyperplanes of S in general position. Letting $H_{i_1', \cdots, i_k} = \{x \mid x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}$ where $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$, this condition means that $S \cap H_{i_1} \cdots \cap H_{i_k} = 0$ or equivalently, $R^n = S \oplus H_{i_1, \cdots, i_k}$ for all choices of $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$. If the k vectors $s_1 = (s_{i1}, \cdots, s_{in})$ $(1 \leqq i \leqq k)$ form a basis for S, then it is easily checked that S is in general position if and only if all k by k minors of (s_{ij}) $(1 \leqq i \leqq k$, $1 \leqq j \leqq n)$ are $\neq 0$. An orthant of R^n is defined to be the set $\{x \mid \epsilon_i x_i > 0, 1 \leqq i \leqq n\}$ where $\{\epsilon_i\}$ denotes any fixed choice of ± 1's. We now prove the following:

THEOREM. *Let S be a k-dimensional subspace of R^n which is in general position. The number of orthants intersected by S is*

$$2 \sum_{i=0}^{k-1} \binom{n-1}{i}.$$

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PROOF. For $k=1$ and $n=k$, the result is obvious so that we may assume $2 \leqq k \leqq n$. Let $0_{n,k}$ = number of orthants intersected by S $(2 \leqq k < n)$. Let $R_{n,k}$ = number of regions into which n hyperplanes of R^k in general position decompose R^k $(n \geqq 1,\ k \geqq 2)$. The orthants of R^n intersected by S stand in 1-1 correspondence with the regions into which the hyperplanes $H_1 \cap S$ $(1 \leqq i \leqq n)$ decompose S. Identifying S with R^k we obtain $0_{n,k} = R_{n,k}$ $(2 \leqq k < n)$. The formula for $R_{n,k}$ is well known and can be found in [1]. We give another derivation of this formula.

Let $L_1, \cdots, L_{n+1}$ be $n+1$ hyperplanes of R^k in general position decomposing R^k $(k \geqq 3)$ into $R_{n+1,k}$ k-dimensional regions. Let L_{n+1} intersect c of these. Each of the regions is divided by L_{n+1} into two regions so that $R_{n+1,k} = R_{n,k} + c$. Let $L_i' = L_i \cap L_{n+1}$ $(1 \leqq i \leqq n)$. Then $L_1', \cdots, L_n'$ are hyperplanes of L_{n+1} in general position decomposing L_{n+1} into $R_{n,k-1}$ $(k-1)$-dimensional regions. These regions are in 1-1 correspondence with the c k-dimensional ones intersected by L_{n+1}. Thus:

$$R_{n+1,k} = R_{n,k} + R_{n,k-1} \qquad (k \geqq 3,\ n \geqq 1). \tag{1}$$

Let $R_k(z) = \sum_{n=1}^{\infty} R_{n,k} z^n$. Since $R_{1,k} = 2$ we conclude from (1) that

$$R_k(z) = R_{k-1}(z)(z/(1-z)) + 2z/(1-z) \qquad (k \geqq 3). \tag{2}$$

Since $R_{n,2} = 2n$ we have $R_2(2) = 2z/(1-z)^2 = 2[z/(1-z) + z^2/(1-z)^2]$. Repeated use of (2) yields

$$R_k(z) = 2[z/(1-z) + \cdots + z^k/(1-z)^k], \tag{3}$$

Equating the nth coefficients of both sides of (3), we obtain the desired formula

$$R_{n,k} = 2 \sum_{i=0}^{k-1} \binom{n-1}{i}.$$

We now obtain Stiemke's theorem as a consequence of the above formula. We may assume that S is in general position; the case where S is not in general position is then treated by a standard limiting argument.

Let $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq j_1 < \cdots < j_{n-k} \leqq n$ denote n integers comprising all integers from 1 to n. $S \oplus H_{i_1 \cdots i_k} = R^n$ as S is in general position. Hence $0 = (S \oplus H_{i_1 \cdots i_k})^\perp = S^\perp \cap H_{i_1}^\perp \cdots_{i_k} = S^\perp \cap H_{j_1 \cdots j_{n-k}}$. Since $S^\perp \cap H_{j_1 \cdots j_{n-k}} = 0$ for all choices of $1 \leqq j < \cdots < j_{n-k} \leqq n$, we conclude that $S^\perp$ is in general position. Thus the number of orthants intersected by $S^\perp$ is $2\sum_{i=0}^{n-1-k} \binom{n-1}{i}$. If x, y are in the same orthant then $x \cdot y > 0$ so that the orthants intersected by S are distinct from those intersected by $S^{\perp}$. Since there are 2^n orthants and

$$2\sum_{i=0}^{k}\binom{n-1}{i}+2\sum_{i=k}^{n-1}\binom{n-1}{i}=2\sum_{i=0}^{n-1}\binom{n-1}{i}=2^n,$$

we conclude that every orthant is intersected by S or $S^{\perp}$. In particular this holds for the positive orthant, thus proving Stiemke's theorem.

References

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