A NECESSARY CONDITION THAT TWO FINITE QUASI-FIELDS COORDINATIZE ISOMORPHIC TRANSLATION PLANES

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Given two quasi-fields it is usually easier to determine whether or not they are isotopic or anti-isotopic than it is to determine whether or not they can coordinatize isomorphic planes. The theorem given here has proved to be quite useful for this purpose. (See, for example, [2].)

A translation plane \( \pi \) may always be coordinatized, in the Hall sense [1], by a right quasi-field (Veblen-Weddenburn system). It is known that two quasi-fields which are either isotopic or anti-isotopic coordinatize isomorphic translation planes. This note gives a necessary condition that two finite quasi-fields which are neither isotopic nor anti-isotopic can coordinatize isomorphic translation planes.

For the remainder of this note let \( F_1, F_2 \) be finite quasi-fields and \( \pi_1, \pi_2 \) the associated translation planes in the sense of Hall [1]. For \( i = 1, 2 \) let \( F_{ip} = \{ a \in F_i \mid a \neq 0 \text{ and } (xy)a = x(ya) \forall x, y \in F_i \} \) = right nucleus of \( F_i \) and let \( F_{im} = \{ a \in F_i \mid a \neq 0 \text{ and } (xa)y = x(ay) \} \) = middle nucleus of \( F_i \). It is easy to see that for each \( a \in F_{ip} \) the mapping \( \beta_a : (x, y) \rightarrow (x, ya), (m) \rightarrow (ma), Y_i = (\infty) \rightarrow Y_i \) is a perspectivity of \( \pi_i \) with center \( Y_i \) and axis the line \( y = 0 \). The correspondence \( a \leftrightarrow \beta_a \) is an isomorphism between \( F_{ip} \) and the set of \( Y_i - O_i, X_i \) perspectivities of \( \pi_i \). Also, for each \( a \in F_{im} \), the mapping \( \gamma_a : (x, y) \rightarrow (xa, y), (m) \rightarrow (mL(a)), Y_i \rightarrow Y_i \) is a \( X_i - O_i, Y_i \) perspectivity of \( \pi_i \) and the correspondence \( a \leftrightarrow \gamma_a \) is an isomorphism between \( F_{im} \) and the set of \( X_i - O_i, Y_i \) perspectivities of \( \pi_i \).

We will use the conventional notation \( Y = (\infty), O = (0, 0), X = (0). \) We will also use the symbol \( |A| \) to denote the cardinality of a set \( A \).

Theorem. Suppose \( F_1 \) and \( F_2 \) are neither isotopic nor anti-isotopic and that \( \pi_1 \) and \( \pi_2 \) are isomorphic. Let \( \sigma \) be an isomorphism from \( \pi_1 \) to \( \pi_2 \) with \( O_1\sigma = O_2, X_1\sigma = X_3, Y_1\sigma = Y_3 \). Let \( D = \{ X_2, Y_2 \} \cap \{ X_3, Y_3 \} \). Assume that if there is an isomorphism from \( \pi_1 \) to \( \pi_2 \) such that \( D \neq \emptyset \) then \( \sigma \) is such an isomorphism. Then

1. If \( |D| = 1 \) then \( (|F_{1p}|, |F_{2p}|) = (|F_{1m}|, |F_{2m}|) = 1. \)

2. If \( |D| = 0 \) then \( (|F_{1p}|, |F_{2p}|) \) and \( (|F_{1m}|, |F_{2m}|) \leq 2. \)

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Proof. Since $F_1$ and $F_2$ are neither isotopic nor anti-isotopic $|D| < 2$. We will prove the theorem for the right nuclei. The proof for the middle nuclei is the same except for the appropriate word replacements. Let $A$ be the set of all $Y_2 - O_2 X_2$ perspectivities of $\pi_2$ and let $B$ be the set of all $Y_3 - O_3 X_3$ perspectivities of $\pi_2$. Then $|A| = |F_{2p}|$ and $|B| = |F_{1p}|$. Let $M = \{ \rho \in L_2 \mid \rho = X_2 \alpha \text{ or } \rho = Y_2 \alpha \text{ for some collineation } \alpha \text{ of } \pi_2 \}$. Then $A$ is a group of permutations on $M - \{ X_2, Y_2 \}$ such that each orbit contains $|F_{2p}|$ elements. Thus $|M| = k |F_{2p}| + 2$ for some integer $k$. Also, $B$ is a group of permutations on $M - \{ X_3, Y_3 \}$ with each orbit containing $|F_{1p}|$ elements so that $|M| = t |F_{1p}| + |D|$ and so $(|F_{1p}|, |F_{2p}|) \leq 2$. If $|D| = 1$ then $(|F_{1p}|, |F_{2p}|) = 1$.

Corollary. In the theorem, if $D = \emptyset$ then the points of $L_\infty$ in $\pi_2$ or $\pi_1$ may be partitioned into three mutually disjoint sets $A_1, A_2, A_3$ such that

- $A_i \equiv 0 \mod|F_{ip}|$ for $i, j = 1, 2, \text{ and } i \neq j$,
- $A_i \equiv 2 \mod|F_{ip}|$ for $i = 1, 2$ and
- $A_3 \equiv 0 \mod(LCM[|F_{1p}|, |F_{2p}|])$.

Proof. Since $D = \emptyset$, $|D| = 0$ so that $|M| \equiv 2 \mod|F_{2p}|$ and $|M| \equiv 0 \mod|F_{1p}|$. Let $M = A_2$. Let $A_1 = \{ \rho \in L_2 \mid \rho = X_3 \alpha \text{ or } \rho = Y_3 \alpha \text{ for some collineation } \alpha \text{ of } \pi_2 \}$. Clearly the proof of the theorem with the appropriate change of symbols shows that $|A_1| \equiv 0 \mod|F_{2p}|$ and $|A_1| \equiv 2 \mod|F_{1p}|$. $D = \emptyset$ implies $A_1 \cap A_2 = \emptyset$.

Let $A_3 = L_2 - (A_1 \cup A_2)$. Then each of $A$ and $B$ acts as a permutation group on $A_3$ with orbits of size $|F_{2p}|$ and $|F_{1p}|$ respectively. Thus, $|A_3| \equiv 0 \mod(LCM[|F_{1p}|, |F_{2p}|])$.

References


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