JORDAN STRUCTURES IN SIMPLE GRADED RINGS

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1. Introduction. In a recent paper [3] we proved graded analogs to theorems of Herstein [1], [2] about the Lie structure of a simple ring. In this note results about the Jordan structure of a simple graded ring will be given. The main results are Theorem 1, which states that a homogeneous Jordan ideal that contains an even element also contains an irrelevant ideal, and Theorem 2, which states that a homogeneous Jordan ideal that is also a subring must contain an irrelevant ideal.

2. Preliminaries. In a graded ring \( R = \bigoplus_{i \in \mathbb{Z}} R_i \), ideals of the form \( \bigoplus_{i \in \mathbb{Z}} R_i \) are irrelevant. \( R \) is a simple graded ring (sgr) if \( R_i R_j \neq (0) \) for all \( i \) and \( j \), and \( R \) has no relevant homogeneous ideals. If \( x \in R \) and \( y \in R \), then \([x, y] = xy - (-1)^{a \beta}yx\) is called their Lie product (Jordan product). The center of \( R \) is \( Z(R) = \{x : [x, y] = 0 \text{ for all } y \in R\} \).

**Proposition 1.** Let \( R = \bigoplus_{i \in \mathbb{Z}} R_i \) be a sgr. If \( \alpha \neq a \in R \), then \( R_i a R_k = R_{i+j+k} \) for all \( i \) and \( k \). If \( b \) is homogeneous and \( R_i b R_0 = (0) \), then \( b = 0 \).

A proof may be found in [3].

3. Lemma. Let \( R \) be a graded ring and let \( U \) be a homogeneous Jordan ideal of \( R \). If \( a, b \in U \) are homogeneous, then for all homogeneous \( x \in R \) we have \([a, b], x \] \( U \).

**Proof.** \((a, [x, b]) - ([a, x], b) = (-1)^{a \beta}[x, (a, b)]\). The left side of the equation is an element of \( U \), so the result follows.

**Theorem 1.** Let \( R = \bigoplus_{i \in \mathbb{Z}} R_i \) be a sgr of characteristic \( \neq 2 \) and let \( U \) be a homogeneous Jordan ideal of \( R \). If \( U \) contains a nonzero even element of \( R \), then \( U \) contains a nonzero irrelevant ideal of \( R \).

**Proof.** Let \( a, b \in U \) be homogeneous. Then \([a, b], x \] \( U \) and \((a, b), x \] \( U \) imply \( 2x(a, b) \in U \) which in turn implies that \( 2x(a, b), y \) \( U \) for all homogeneous \( y \). Thus, \( 2R_i(a, b)R_j \subseteq U \) for all \( i \) and \( j \).

If \( 2R_i(a, b)R_j = (0) \) for all \( i \) and \( j \), then by Proposition 1 \( (a, b) = 0 \), and so in this case \((U, U) = (0) \). If \( 0 \neq a \in U \) is even, \( 0 = (a, (a, x)) \) implies \( 2axa = 0 \) for all homogeneous \( x \). Thus, \( a = 0 \), a contradiction.

Hence, there exist \( i \) and \( j \) such that \( 0 \neq 2R_i(a, b)R_j \subseteq U \). Therefore, \( U \supseteq \bigoplus_{k \geq a+b+i+j} R_k \neq (0) \).

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Proposition 2. Let \( R \) be a sgr, \( \text{char } R \neq 2 \), and let \( U \) be a homogeneous Jordan ideal of \( R \) that does not contain a nonzero irrelevant ideal of \( R \). If \( a \in U \) satisfies \( [a, R] \subseteq U \), then \( a = 0 \).

Proof. Let \( x \) and \( y \) be homogeneous. If \( a \neq 0 \), then \( [a, x] \in U \) and \( (a, x) \in U \) imply \( ax \in U \), so \( (ax, y) \in U \). Thus, \( yax \in U \), so \( U \supseteq \oplus_{i \in a} R_i \).

Corollary. With \( R \) and \( U \) as above, \( U \cap Z(R) = (0) \).

Theorem 2. Let \( R \) be a sgr, \( \text{char } R \neq 2 \), and let \( U \) be a homogeneous Jordan ideal and a subring of \( R \). Either \( U = (0) \) or \( U \) contains a nonzero irrelevant ideal of \( R \).

Proof. If \( (U, U) = (0) \), \( a = 0 \) for all even \( a \in U \). If \( 0 \neq a \in U \) is odd, \( a^2 = 0 \) and so \( a(a, x) = 0 \) for all even \( x \in R \). Hence, \( axa = 0 \), so \( a = 0 \). Thus, \( (U, U) = (0) \) implies that \( U = (0) \).

If \( (U, U) \neq (0) \) let \( a \) and \( b \in U \), \( (a, b) \neq 0 \), and let \( c \) be homogeneous. Then \( (ab, c) = (a, b)c + (-1)^{ab}(a, c) + (-1)^{a(b+c)}(b, ca) \). Thus, \( (a, b)R_i \subseteq U \) for all \( i \). Let \( d \) be homogeneous. Then \( (d, (a, b)c) = d(a, b)c + (-1)^{d(a+b+c)}(a, b)cd \subseteq U \). Now,

\[
(a, bcd) = abcd + (-1)^{e(b+c+d)}bcd \in U.
\]

An examination of the parities involved shows that \( (a, bcd) = (a, b)cd \pm (ba, cd) \pm (cd, b)a \). Thus \( (a, b)cd \subseteq U \), and so \( d(a, b)cd \subseteq U \). This implies that \( U \supseteq \oplus_{i \in a+b} R_i \).

References


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