ON UNIQUENESS OF QUASI-SPLIT REAL SEMISIMPLE LIE ALGEBRAS

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Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{g}_0$ a real form of $\mathfrak{g}$, i.e. $\mathfrak{g}=\mathfrak{g}_0+\sqrt{(-1)}\mathfrak{g}_0$. The form $\mathfrak{g}_0$ is called *quasi-split* if there is a subalgebra $\mathfrak{b}\subset\mathfrak{g}_0$ such that $\mathfrak{b}+\sqrt{-1}\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_0$ is said to be *full rank* if rank $\mathfrak{g}=$rank $\mathfrak{k}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}_0$. The purpose of this note is to show that these two properties characterize a unique real form of $\mathfrak{g}$.

THEOREM. *Every complex semisimple Lie algebra $\mathfrak{g}$ contains a full rank, quasi-split real form, which is unique up to isomorphism.*

PROOF. We first show existence. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$' and Δ a set of roots with respect to $\mathfrak{h}$. If $\mathfrak{h}_0$ is the real subspace of $\mathfrak{h}$ spanned by Δ, then a real form $\mathfrak{g}_0$ is defined, up to isomorphism, by the choice of a complex-conjugate involution σ of $\mathfrak{g}$ which leaves $\mathfrak{h}_0$ invariant. (The reader is referred to [**1**, Chapter IX] for proof of this and other facts about involutions and the classification of real forms, and to [**2**, Chapter V] for known facts about root systems and Weyl groups.) We write $\sigma|\mathfrak{h}_0$ for the restriction of σ to $\mathfrak{h}_0$. $\mathfrak{g}_0$ is quasi-split iff $\sigma|\mathfrak{h}_0$ leaves invariant an open chamber of the Weyl group W acting on $\mathfrak{h}_0$. Let Π be a set of simple positive roots for Δ, and let τ be the unique element of W which maps Π to $-\Pi$. Then $-\tau$ leaves the fundamental chamber invariant. Set $\sigma|\mathfrak{h}_0=-\tau$ and extend to a complex-conjugate involution σ of $\mathfrak{g}$. Then the real form $\mathfrak{g}_0=\{x\in\mathfrak{g}\,|\,\sigma x=x\}$ is quasi-split.

We claim $\mathfrak{g}_0$ is full rank. To see this, let $\mathfrak{h}_0^{+1}$, $\mathfrak{h}_0^{-1}$ be respectively the $+1$ and -1 eigenspaces of σ in $\mathfrak{h}_0$. Put $i=\sqrt{-1}$. Then $\mathfrak{h}_0^{+1}+i\mathfrak{h}_0^{-1}$ is a Cartan subalgebra of $\mathfrak{g}_0$, and $i\mathfrak{h}_0^{+1}+i\mathfrak{h}_0^{-1}$ is a Cartan subalgebra of the associated compact form. One knows [**1**, p. 335, Theorem 3.2] that $\mathfrak{g}_0$ is full rank iff the involution of $\mathfrak{h}$ defined by $\mu|i\mathfrak{h}_0^{-1}=1$, $\mu|i\mathfrak{h}_0^{+1}=-1$ is inner. But this follows immediately from [**1**, Proposition 2.5] since $\tau\in W$, and the linear extension of τ to all of $\mathfrak{h}$ agrees with μ on $i\mathfrak{h}_0^{+1}+i\mathfrak{h}_0^{-1}$.

To prove uniqueness, suppose $\mathfrak{g}_0'$ is another quasi-split, full rank real form defined by an involution σ' which leaves $\mathfrak{h}_0$ invariant. Then since $\mathfrak{g}_0'$ is quasi-split $\sigma|\mathfrak{h}_0$ leaves invariant a chamber of W in $\mathfrak{h}_0$, and

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we may assume, up to conjugation of σ, that it is the fundamental chamber. By the above, since $\mathfrak{g}_0$ is full rank $\sigma | i\mathfrak{h}_0^{+1} + i\mathfrak{h}_0^{-1}$ (extended linearly to $\mathfrak{h}_0$) is in W, and hence $-\sigma | \mathfrak{h}_0 = \tau$, since τ is the unique element of the Weyl group which maps Π to $-\Pi$. We now show that there exists η, an inner automorphism of $\mathfrak{g}$, such that $\sigma' = \eta^{-1}\sigma\eta$. Let $\{X_\alpha | \alpha \in \Pi\}$ be the simple root vectors of a Weyl basis of $\mathfrak{g}$. Then σ and σ' are completely determined by the complex numbers c_α and c_α', where $\sigma X_\alpha = c_\alpha X_\alpha$ and $\sigma' X_\alpha = c_\alpha' X_\alpha$. Let η be the linear automorphism defined by $\eta | \mathfrak{h}$ is the identity, and $\eta X_\alpha = \sqrt{c_\alpha'} / \sqrt{c_\alpha} X_\alpha$ for α simple. Then we claim $\eta^{-1}\sigma\eta X_\alpha = \sigma' X_\alpha$. Indeed,

$$\eta^{-1}\sigma\eta X_\alpha = \eta^{-1}\sigma\sqrt{c_\alpha'} / \sqrt{c_\alpha} X_\alpha = \eta^{-1}\sqrt{(c_\alpha c_\alpha')} X_{\sigma\alpha}.$$

Since σ permutes the positive simple roots, $\sigma\alpha$ is again simple, and

$$\eta^{-1}\sqrt{(c_\alpha c_\alpha')} X_{\sigma\alpha} = \sqrt{(c_\alpha c_{\sigma\alpha})}\sqrt{(c_\alpha' / c_{\alpha\sigma}')} X_{\sigma\alpha}.$$

But $\sigma^2 = 1$ and similarly for σ'^2. Therefore

$$\sqrt{(c_\alpha c_{\sigma\alpha})}\sqrt{(c_\alpha' / c_{\sigma\alpha}')} X_{\sigma\alpha} = \sqrt{(c_\alpha' / c'_{\sigma\alpha})} X_{\sigma\alpha} = c_\alpha' X_{\sigma\alpha}$$

which proves that $\eta^{-1}\sigma\eta = \sigma'$. Hence $\mathfrak{g}_0'$ is isomorphic to $\mathfrak{g}_0$.

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REMARK. One may construct a full rank quasi-split $\mathfrak{g}_0$ as follows. Any $\phi \in \Delta^+ = \{\phi \in \Delta | \phi > 0\}$ can be written $\phi = \sum_{\alpha\in\pi} n_\alpha \alpha$, where n_α is a nonnegative integer [1, p. 331, Theorem 2.2]. Write $\mathrm{ht}(\phi) = \sum_{\alpha\in\pi} n_\alpha$, the *height of* ϕ. Let

$$\Delta_1 = \{\phi \in \Delta^+ | \mathrm{ht}\ (\phi) \text{ is even}\}$$

and

$$\Delta_2 = \{\phi \in \Delta^+ | \mathrm{ht}\ (\phi) \text{ is odd}\}.$$

Then $\mathfrak{g}_0 = i\mathfrak{h}_0 + \sum_{\phi\in\Delta_1} R(X_\phi - X_{-\phi}) + \sum_{\phi\in\Delta_1} iR(X_\phi + X_{-\phi}) + \sum_{\psi\in\Delta_2} R(X_\psi - X_{-\psi}) + \sum_{\psi\in\Delta_2} iR(X_\psi - X_{-\psi})$ is full rank, quasi-split, where $R(X)$ denotes the real subspace spanned by $X \in \mathfrak{g}$. Note that

$$\mathfrak{k} = i\mathfrak{h}_0 + \sum_{\phi\in\Delta^\cdot} R(X_\phi - X_{-\phi}) + \sum_{\phi\in\Delta_1} iR(X_\phi + X_{-\phi})$$

is a maximal compact subalgebra, and Δ_1 is a root system for $\mathfrak{k}$. Hence $\mathfrak{g}_0$ is completely determined by the condition that no simple root be in Δ_1.

We list here the full rank, quasi-split real forms for the complex simple Lie algebras, using the notation of [1, p. 354].

	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{k}$
$\mathfrak{a}_{2q-1}$	$\mathfrak{sl}(2q, \boldsymbol{C})$	$\mathfrak{su}(q, q)$	$\mathfrak{su}(q)\oplus\mathfrak{su}(q)\oplus\boldsymbol{R}$
$\mathfrak{a}_{2q}$	$\mathfrak{sl}(2q+1, \boldsymbol{C})$	$\mathfrak{su}(q+1, q)$	$\mathfrak{su}(q+1)\oplus\mathfrak{su}(q)\oplus\boldsymbol{R}$
$\mathfrak{b}_q$	$\mathfrak{so}(2q+1, \boldsymbol{C})$	$\mathfrak{so}(q+1, q)$	$\mathfrak{so}(q+1)\oplus\mathfrak{so}(q)$
$\mathfrak{c}_q$	$\mathfrak{sp}(q, \boldsymbol{C})$	$\mathfrak{sp}(q, \boldsymbol{R})$	$\mathfrak{u}(q)$
$\mathfrak{d}_{2q}$	$\mathfrak{so}(4q, \boldsymbol{C})$	$\mathfrak{so}(2q, 2q)$	$\mathfrak{so}(2q)\oplus\mathfrak{so}(2q)$
$\mathfrak{d}_{2q+1}$	$\mathfrak{so}(4q+2, \boldsymbol{C})$	$\mathfrak{so}(2q+2, 2q)$	$\mathfrak{so}(2q+2)\oplus\mathfrak{so}(2q)$
$\mathfrak{e}_6$		$E\,\mathrm{II}$	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$
$\mathfrak{e}_7$		$E\,\mathrm{V}$	$\mathfrak{su}(8)$
$\mathfrak{e}_8$		$E\,\mathrm{VIII}$	$\mathfrak{so}(16)$
$\mathfrak{f}_4$		$E\,\mathrm{I}$	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$
$\mathfrak{g}$		$\boldsymbol{G}$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$

References

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2. J.-P. Serre, *Algebres de Lie semi-simples complexes*, Benjamin, New York, 1966. MR **35** #6721.

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