FATOU'S LEMMA IN SEVERAL DIMENSIONS

DAVID SCHMEIDLER

ABSTRACT. In this note the following generalization of Fatou's lemma is proved:

**Lemma.** Let \( (f_n)_{n=1}^{\infty} \) be a sequence of integrable functions on a measure space \( S \) with values in \( \mathbb{R}^d \), the nonnegative orthant of a \( d \)-dimensional Euclidean space, for which \( \int f_n \to a \in R^d_+ \). Then there exists an integrable function \( f \) from \( S \) to \( \mathbb{R}^d_+ \), such that a.e. \( f(s) \) is a limit point of \( (f_n(s))_{n=1}^{\infty} \) and \( \int f \leq a \).

1. Introduction. When \( d = 1 \), the result is a form of Fatou's lemma. The assertion above is applied in mathematical economics [4]. It is also strongly connected with the theory of set valued functions [2] or correspondences [3]. The nontrivial part of the arguments is limited to the case where \( S \) is an atomless measure space. In the purely atomic case the lemma is reduced to a simple exercise in series. In any case, the lemma cannot be proved by a successive application of Fatou's lemma \( d \) times.

A few corollaries of the lemma are proved in §3.

2. Preliminary results and the proof of the lemma. Let \( (A_n)_{n=1}^{\infty} \) be a sequence of (nonempty) subsets of \( \mathbb{R}^d \). We denote by \( \text{Lim Sup}_{n=1}^{\infty} A_n \) the set of all the limit points of the sequences \( (a_n)_{n=1}^{\infty} \) with \( a_n \in A_n \), \( n = 1, 2, \cdots \). Denote by \( x \cdot y \) the inner product, \( \sum_{i=1}^{d} x_i y_i \), in \( \mathbb{R}^d \).

**Proposition 1.** For each \( p > 0 \) there is an integrable function \( g \) such that \( p \cdot f \leq p \cdot a \) and a.e. \( g(s) \in \text{Lim Sup}_{n=1}^{\infty} \{ f_n(s) \} \) and \( p \cdot g(s) = \lim \inf f_n \cdot f_n(s) \).

Proof. Define \( h(s) = \lim \inf f_n \cdot f_n(s) \). As \( f_n(s) \to f(a) \), by Fatou's lemma \( f h \leq p \cdot a \). Now we decompose \( h \) to \( d \) integrable components \( g^1, \cdots, g^d \) such that a.e. \( p \cdot g(s) = h(s) \).

Define:

\[
g_n(s) = \inf \{ f_k(s) \mid k \geq n \ \text{and} \ p \cdot f_k(s) < h(s) + 1/n \}.
\]

For each \( r \in \mathbb{R}^d_+ \) and \( n = 1, 2, \cdots \) one has

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\{s \mid g_n^i(s) < r\} = \bigcup_{k \geq n} \{s \mid f_k^i(s) < r\} \cap \{s \mid p \cdot f_k(s) < h(s) + 1/n\}.

Hence \((g_n^i)\) is a monotone sequence of measurable functions, each bounded by the integrable function \((1/p^i)(h+1)\). Define \(g^i(s) = \lim_n g_n^i(s)\) then \(g^i\) is an integrable function and a.e. \(p^i g^i(s) \leq h(s)\) and \(g^i(s) \in \text{Lim Sup}_n \{f_n^i(s)\}\).

Proceed by induction:

\[
\begin{align*}
g_n^i(s) &= \inf \left\{ f_k^i(s) \mid k \geq n \text{ and } p \cdot f_k(s) < h(s) + 1/n \right\} \\
\text{and } f_k^i(s) &< g^j(s) + 1/n, j = 1, \ldots, i - 1.
\end{align*}
\]

It is easy to check that \(g_n^i(s)\) is well defined and \(g^i(s) = \lim_n g_n^i(s)\) is an integrable function with \(\sum_{j=1}^i p^j g^j(s) \leq h(s)\) and \((g^i(s), \ldots, g^i(s)) \in \text{Lim Sup}_n \{f_n^i(s)\}\) a.e. After \(d\) steps we have \(g(s) = (g^1(s), \ldots, g^d(s))\) such that a.e. \(p \cdot g(s) = h(s)\) and \(g(s) \in \text{Lim Sup}_n \{f_n(s)\}\). Q.E.D.

Denote: \(Q_y = \{x \in R^d_+ \mid x^y\}, y \in R^d_+.\)

**Proposition 2.** Let \(A\) be a closed, convex subset of \(R^d_+\) and \(y \in R^d_+\) such that \(AC\cap Q_y = \emptyset\). Then there is \(q > 0\) with

\[
\sup \{q \cdot x \mid x \in Q_y\} < \inf \{q \cdot x \mid x \in A\}.
\]

**Proof.** By the separation theorem there are \(p\) and \(\alpha\) such that \(p \cdot x < \alpha < p \cdot z\) for all \(x \in Q_y\) and all \(z \in A\). Let \(p'\) be the vector obtained from \(p\) by substitution of zero for each negative coordinate of \(p\). For \(x \in A, x \geq 0\) so \(p' \cdot x \geq p \cdot x > \alpha\). For \(x \in Q_y\) let \(x'\) denote the vector obtained from \(x\) by substitution of zero for those coordinates which we changed previously in \(p\). Of course, \(x' \in Q_y\), so \(p' \cdot x = p \cdot x' < \alpha\). Denote by \(p_\delta\) the vector obtained from \(p'\) by substitution of \(\delta > 0\) for each zero in \(p'\). Again for each \(x \in A\), \(p_\delta \cdot x \geq p' \cdot x > \alpha\). For \(x \in Q_y\) one has \(p_\delta \cdot x \leq p' \cdot x + \delta \alpha < \alpha + \delta \alpha\). Because of the compactness of \(Q_y\) there is a \(\delta' > 0\) such that \(q = p_{\delta'}\) fulfills the requirements of the proposition. Q.E.D.

For each \(s\) in \(S\) let \(F(s)\) be a nonempty subset of \(R^d\).

Following [2] we define:

\[
\int F = \left\{ \int h \mid h \text{ is integrable and a.e. } h(s) \in F(s) \right\}
\]

**Proposition 3.** Let \(A = \int \text{Lim Sup}_n \{f_n(s)\}\) and \(q > 0\) such that for each \(x \in A\), \(q \cdot x \geq q \cdot a\). Then there is a subsequence \((f_{n_k})\) of \((f_n)\) such that for each \(x \in \int \text{Lim Sup}_k \{f_{n_k}(s)\}, q \cdot x = q \cdot a\).
PROOF. Denote \( h_n(s) = \inf \{ q \cdot f_k(s) \mid k \geq n \} \) and \( h(s) = \lim_n h_n(s) \) \( = \lim \inf_n q \cdot f_n(s) \). Using Proposition 1 for \( p = q \) one has: \( \int h = q \cdot \int g \leq q \cdot a \) and \( \int g \in A \). By the condition of the proposition \( q \cdot \int g \geq q \cdot a \), so \( \int h = q \cdot a \). For each \( s \in S \), \( h_n(s) \leq q \cdot f_n(s) \) so \( \int q \cdot f_n - h_n = \int (q \cdot f_n - h_n) = \int q \cdot f_n - \int h_n - q \cdot a - \int h = 0 \). Also as \( \int h - h_n \to 0 \), we get \( \int q \cdot f_n - h_n \to 0 \). Convergence in the mean implies the convergence of a subsequence a.e. Hence there is a subsequence \( (f_{n_k}) \) such that a.e. \( q \cdot f_{n_k}(s) \to h(s) \). Consequently for a.e. \( s \in S \) and each \( x \in \text{Lim Sup}_k \{ f_{n_k}(s) \} \), \( q \cdot x = h(s) \). Integrating over \( S \) completes the proof. Q.E.D.

PROPOSITION 4. Let \( S \) be atomless and for each \( s \in S \) let \( F(s) \) be a non-empty subset of \( R^d \). Then \( \int F \) is convex.

PROOF. This is an elementary theorem about integrals of correspondences due to Richter, [5]. (The proof appears also in [3], p. 369.) This theorem is a simple consequence of Lyapunov convexity theorem and will not be reproved here.

PROPOSITION 5. Let \( a_{k,n} \in R^d_+ \) for \( k, n = 1, 2, \ldots \) and assume that \( \sum_{k=1}^\infty a_{k,n} \to a \) (where \( n \to \infty \)). Then there is a sequence \( (b_k)_{k=1}^\infty \) such that \( \sum_{k=1}^\infty b_k \leq a \) and for each \( k, b_k \in \text{Lim Sup}_n \{ a_{n,k} \} \). Moreover, if there is in addition, a sequence \( (c_k)_{k=1}^\infty \) such that for each \( n \) and \( k, a_{n,k} \leq c_k \) and \( \sum_{k=1}^\infty c_k = c \in R^d_+ \), then \( \sum_{k=1}^\infty b_k = a \).

REMARK. The first part of this proposition is exactly the statement of the lemma in case of a purely atomic measure space; the second part is related to Corollary 1 in §3.

PROOF. Reasoning by compactness, the sequence of sequences \( (a_{k,n})_{k=1}^\infty \) has a pointwise converging subsequence \( (a_{k,n})_{k=1}^\infty \), the limit of which we denote by \( (b_k)_{k=1}^\infty \). Thus, for each \( k, b_k = \lim_k a_{k,n} \), i.e. \( b_k \in \text{Lim Sup}_n \{ a_{n,k} \} \). We have to prove that \( \sum_{k=1}^\infty b_k \leq a \). Assume the contrary, i.e. there is a coordinate \( i \), an integer \( N \) and a number \( \epsilon > 0 \) such that \( \sum_{k=1}^N b_k \geq a^i + \epsilon \). For each \( k \) let \( M_k \) be such that \( n_j > M_k \) imply \( b_k < a_{k,n_j}^i + \epsilon / 2N \), and let \( M_0 \) be such that \( n > M_0 \) imply \( a_{k,n}^i < a^i + \epsilon / 4 \). Define \( M = \max \{ M_0, M_1, \ldots, M_N \} \). Then for \( n_m > M \) one has: \( \sum_{k=1}^{\infty} a_{k,n_m}^i \geq \sum_{k=1}^N a_{k,n_m}^i > \sum_{k=1}^N b_k^i - \epsilon / 2 \geq a^i + \epsilon / 2 \), a contradiction.

Now assume the additional condition and apply the first part of the proposition to the sequence \( (c_k = a_{k,n_j})_{k=1}^\infty \). Q.E.D.

A point \( x \) in a set \( B \) in \( R^d \) is called admissible if \( x \geq y \in B \) imply \( x = y \). If for \( x \in B \) there exists a vector \( p > 0 \) such that for each \( y \in B \), \( p \cdot x \leq p \cdot y \) then \( x \) is called strictly admissible. Of course, a strictly admissible point of a set is also admissible.
Proposition 6. The admissible points of a closed convex set in $\mathbb{R}^d$ belong to the closure of the strictly admissible points of this set.

Proof. This is a theorem of Arrow-Barankin-Blackwell, \cite{1}. (I thank G. Debreu for this reference.)

Proposition 7. Let $S$ be atomless and set $A = \operatorname{fLim Sup} \{f_n(s)\}$. Then $A$ is convex and $A \cap Q_a \neq \emptyset$.

Proof. The convexity of $A$ is implied by Proposition 4. $A$ is nonempty by Proposition 1. Assume that $A \cap Q_a = \emptyset$. By Proposition 2 there is a vector $q > 0$ with

$$\inf\{q \cdot x | x \in A\} > q \cdot a \quad (q \cdot a = \sup\{q \cdot x | x \in Q_a\}).$$

The last inequality contradicts Proposition 1. Q.E.D.

Proof of the lemma. We decompose $S$ to an atomless part and a purely atomic part. The lemma can be proved separately for each part. Proposition 5, as remarked above, proves the lemma for the purely atomic case. (One can assume, without a loss of generality, that in $S$ there are at most $\aleph_0$ atoms.)

Now assume that $S$ is atomless. We prove the lemma reasoning by induction on $\dim (A)$. ($A$ denotes the $\operatorname{fLim Sup} \{f_n(s)\}$ and $\dim (A)$ is the linear dimension of the smallest flat containing $A$.) By Proposition 7, $\dim (A) \geq 0$ and if $\dim (A) = 0$ then the lemma holds. Given $0 < l \leq d$ assume that the lemma holds when $\dim (A) < l$ and we shall prove it for the case $\dim (A) = l$. The induction hypothesis states that for each sequence of integrable functions $g_n : S \to \mathbb{R}^d$ with $\int g_n \to b$ and $\dim (\operatorname{fLim Sup} \{g_n(s)\}) < l$ one has

$$\int \operatorname{Lim Sup} \{g_n(s)\} \cap Q_b \neq \emptyset.$$

In view of Proposition 7 it is sufficient to prove that the admissible points of $A$ belong to $A$.

Claim 1. The strictly admissible points of $A$ belong to $A$.

Let $b \in A$ and $q > 0$ such that $q \cdot b \geq q \cdot x$ for each $x \in A$. If $b \in \text{rel-int } A$ then $b \in A$ because of the convexity of $A$. In the other case the origin is a boundary point of $A - \{b\}$ in the subspace $H - \{b\}$ of $\mathbb{R}^d$, where $H$ is the smallest flat containing $A$. Then there is $q' \neq 0$ in $H - \{b\}$ for which $q' \cdot x \geq 0$ for each $x \in A - \{b\}$ and for at least one point of $A - \{b\}$, say $x_0$, $q' \cdot x_0 > 0$. Hence there is $\epsilon > 0$ such that defining $\rho = q' + (1 - \epsilon)q$ we have: $\rho > 0$, $\forall x \in A$, $\rho \cdot x \geq \rho \cdot b$ and a strict inequality for at least one point of $A$. So

$$\dim (A \cap \{x | \rho \cdot x = \rho \cdot b\}) < l.$$

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Let $y_n \to b$ with $y_n \in A$ for $n = 1, 2, \cdots$. Hence there is a sequence of integrable functions $(g_n)$ such that for each $n$, $\int g_n = y_n$ and a.e. $g_n(s) \in \operatorname{Lim Sup}_k \{f_k(s)\}$. Define $B = \int \operatorname{Lim Sup}_n \{g_n(s)\}$, then $B \subset A$ because a.e. $\operatorname{Lim Sup}_n \{g_n(s)\} \subset \operatorname{Lim Sup}_n \{f_n(s)\}$. In consequence $p \cdot x \geq p \cdot b$ for each $x \in B$ and $\dim (B \cap \{x \mid p \cdot x = p \cdot b\}) < l$. Now the conditions of Proposition 3 are fulfilled for $(g_n)$, $B$, $b$ and $p$, hence there is a subsequence $(g_{n_k})_{k=1}^{\infty}$ such that
\[ \int \operatorname{Lim Sup}_k \{g_{n_k}(s)\} \subset B \cap \{x \mid p \cdot x = p \cdot b\}. \]

The induction hypothesis completes the proof of the claim.

Claim 2. The admissible points of $\overline{A}$ belong to $A$.

Denote by $b$ an admissible point of $\overline{A}$. Because of Proposition 6 and Claim 1 there is a sequence $(y_n)$ of strictly admissible points in $A$ such that $y_n \to b$. Set $(g_n)$, $B$ and $H$ as in the proof of Claim 1. For each $n$ there is a vector $q_n$ with $q_n \cdot q_n = 1$ and $q_n \cdot x \geq q_n \cdot y_n$ for each $x \in A$. We may assume, in addition that for each $n$, $q_n \in H - \{y_n\}$. (Note that $H - \{y_n\} = H - \{b\}$ for each $n$.) Otherwise, we have for some $n$, that 0 is an interior point of $A - \{y_n\}$ in $H - \{y_n\}$, or equivalently: $y_n \in \operatorname{rel-int} A$. But then, because $y_n$ is a strictly admissible point of $\overline{A}$, it implies that each point of $\overline{A}$ is strictly admissible and in this case Claim 2 is a consequence of Claim 1. Thus $(q_n)$ has a limit point $q$ in $H - \{b\}$. Assume, without loss of generality, that $q_n \to q$. For each $x \in A - \{b\}$, $q \cdot x \geq 0$ and $\dim (\{x \in H - \{b\} \mid q \cdot x = 0\}) < l$. Hence, in order to complete the proof of the claim by the induction hypothesis, it is sufficient to show that for each $x \in B$, $q \cdot x = q \cdot b$.

Assume, per absurdum, that there is $x_0 \in B$ with $q \cdot x_0 > q \cdot b$. Because $b \in \overline{A}$ there is $z \in B$ with $q \cdot x_0 > q \cdot z$. Let $h$ and $g$ be two integrable functions such that: $x_0 = \int g$, $z = \int h$ and a.e. $g(s) \in \operatorname{Lim Sup}_n \{g_n(s)\}$ and $h(s) \in \operatorname{Lim Sup}_n \{f_n(s)\}$. As a consequence of the last inequality there is a nonnull set $U$ defined:
\[ U = \{s \in S \mid q \cdot h(s) < q \cdot g(s)\}. \]

Consequently, for each $s \in U$ there are $N(s)$ and $\epsilon(s) > 0$ such that for each $n > N(s)$, $q_n \cdot h(s) < q_n \cdot g(s) + \epsilon(s)$. Because $g(s) \in \operatorname{Lim Sup}_n \{g_n(s)\}$ there is $n(s) > N(s)$, $s \in U$, such that $q_n(s) \cdot h(s) < q_n(s) \cdot g_n(s)$. Since $U = \bigcup_{s=1}^{\infty} \{s \in U \mid n(s) = k\}$, there are $k$ and a nonnull subset, $V$, of $U$ such that for each $s \in V$, $q_k \cdot h(s) < q_k \cdot g_k(s)$. Define a function $\tilde{g}$ by: $\tilde{g}(s) = h(s)$ for $s \in V$ and $\tilde{g}(s) = g_k(s)$ for $s \notin V$. Then a.e. $\tilde{g}(s) \in \operatorname{Lim Sup}_n \{f_n(s)\}$ and $q_k \cdot \int \tilde{g} < q_k \cdot \int g_k = q_k \cdot y_k$—a contradiction. Q.E.D.
3. **Corollaries.** The first two corollaries were proved by Aumann [2], (with some restrictions on $S$). Our proof, based on the lemma, is shorter and simpler than his direct proof. These corollaries have direct application in mathematical economy [4], [6]. As to Corollary 4, it is natural to assume that it has a direct elementary proof but not as short as the one below.

**Corollary 1.** Let $(f_n)$ be a sequence of integrable functions from $S$ to $\mathbb{R}^d$ such that $\int f_n \to a$ and there is an integrable function $g$ with $|f_n(s)| \leq |g(s)|$ for a.e. $s \in S$ and $n = 1, 2, \ldots$. Then there is an integrable function $f$ such that $\int f = a$ and a.e. $f(s)$ is a limit point of $(f_n(s))$.

**Proof.** As in the proof of the lemma, we can deal separately with each of the two cases: $S$ is atomless, $S$ is purely atomic. In the second case, Proposition 5 proves the corollary. We assume, for the rest of the proof, that $S$ is atomless.

Let $d_i; i = 1, \ldots, 2^d$ be the vectors in $\mathbb{R}^d$ with coordinates 1 or $-1$. Define $d_iVx$ to be the vector in $\mathbb{R}^d$ the $j$th coordinate of which is $d_{i}(x)^j$ and define $e = (1, \ldots, 1)$. Then for each $n$ and a.e. $s$, $d_i\nabla f_n(s) \leq e|g(s)|$, $i = 1, \ldots, 2^d$. Now apply the lemma to $(e|g| + d_i\nabla f_n)_{n=1}^\infty$. For each $i$ we get an integrable function $f_i$, such that $d_iVf_i \leq d_iVg$ and a.e. $h_i(s) \in \text{Lim Sup}_n \{f_n(s)\}$. So, using Proposition 4, we get: $a \in \text{Lim Sup}_n \{f_n(s)\}$. Q.E.D.

**Corollary 2.** For each $s$ let $(F_n(s))$ be a sequence of nonempty subsets of $\mathbb{R}^d$ with the property: $x \in F_n(s)$ imply $|x| \leq g(s)$, for some integrable function $g$. Then

$$
\text{Lim Sup}_n \int F_n \subset \int \text{Lim Sup}_n F_n(s).
$$

**Proof.** Assume that $x \in \text{Lim Sup}_n \int F_n$. Then $x$ is a limit point of a sequence $(x_n)$ with $x_n \in \int F_n$. To simplicate notation assume that $x_n \to x$. $x_n \in \int F_n$ means that $x_n = \int f_n$ for an integrable function $f_n$ with $f_n(s) \in F_n(s)$ a.e. By Corollary 1 there is an integrable function $f$ with $\int f = x$ and a.e. $f(s) \in \text{Lim Sup}_n \{f_n(s)\}$. Hence we completed the proof since a.e. $\text{Lim Sup}_n \{f_n(s)\} \subset \text{Lim Sup}_n F_n(s)$. Q.E.D.

**Corollary 3.** Let $F$ be a closed-valued correspondence from $S$ to $\mathbb{R}^d_+$ i.e. $F(s)$ is a nonempty, closed subset of $\mathbb{R}^d_+$, for each $s \in S$. Then $\int F$ contains all the admissible points of its closure.

**Proof.** Let $x$ be an admissible point of $\int F$. Then there is a sequence $x_n \to x$ with $x_n \in \int F$ for each $n$. It means that there is a sequence $(f_n)$ of integrable functions with $f_n(s) \in F(s)$ a.e. for $n = 1, 2, \ldots$. By the
lemma there is an integrable function $f$ with $\int f \leq x$ and a.e. $f(s) \in \text{Lim Sup}_n \{f_n(s)\} \subset F(s)$, where the inclusion is implied by the condition that $F(s)$ is closed for each $s$. Hence $\int f \in \int F$ and because $x$ is an admissible point, we get $\int f = x \in \int F$. Q.E.D.

**Corollary 4.** Let $A$ be a closed set in $\mathbb{R}^d$. Then $\text{conv} (A)$ contains all the admissible points of its closure.

**Proof.** Let $S$ be an atomless probability measure space. (The word “Probability” means that the measure of $S$ is 1.) Define $F(s) = A$ for each $s \in S$. Then, by Corollary 3, the following claim completes the proof.

**Claim 3.** Let $S$ be an atomless probability measure space and $A$ in $\mathbb{R}^d$. Then $\text{conv} (A) = \int F$, where $F(s) = A$ for each $s \in S$.

By Proposition 4, $\text{conv} (A) \subset \int F$. The other inclusion can be easily proved by induction of the dimension and is left to the reader. Q.E.D.

**References**


Hebrew University of Jerusalem