A SEQUENTIALLY CLOSED COUNTABLE DENSE SUBSET OF $I^I$

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Let $I$ denote the closed unit interval, $N$ the positive integers, and $R$ the reals. By $I^I$ we mean the collection of all functions $x = x(t)$ from $I$ into $I$ with the product topology. A sequentially closed set in $I^I$, i.e., a set containing all its sequential limits, is not in general closed, $I^I$ not being first countable. In fact, there are proper subsets of $I^I$ that are sequentially closed and dense, e.g.,

$$\{ x \in I^I \mid x \text{ is Lebesgue measurable} \}$$

and

$$\{ x \in I^I \mid x^{-1}(I \setminus \{0\}) \text{ is countable} \}.$$ 

These sets have cardinality $2^c$ and $c$, respectively. In [5], we raised the question of the existence of a countable such set. We here answer this question affirmatively by showing that $I^I$ contains a countable dense subset that contains no nontrivial convergent sequences at all. Separability of $I^I$ is well known, and follows, for instance, from a more general result of Pondiczery [4] (see also Marczewski [3]).

Let $J \subset R$ be a maximal subset of irrationals linearly independent over the integers. Since $I$ and $I$ have the same cardinality, $I^I$ and $I'$ are homeomorphic, and so our goal will be reached by exhibiting a countable dense subset of $I'$ having no nontrivial convergent sequences in it. Let $S$ be the set $\{ x_n \}_{n \in N}$ in $I'$ such that

$$x_n(t) = (nt) \text{ for all } t \in J,$$

where $(r)$, for $r \in R$, denotes the number in $[0, 1)$ congruent to $r$ modulo 1. The fact that $S$ is dense in $I'$ follows directly from Kronecker's well-known theorem regarding the simultaneous approximation mod 1 of arbitrary real numbers by integral multiples of linearly independent irrationals [2, Chapter XXIII].

Now consider any convergent sequence $\{ x_{n_k} \}$ of elements of $S$. Then $\{ x_{n_k}(t) \}$ converges for each $t \in J$. For $t \in R$, there exist, by the maximality of $J$, finitely many integers $m_i$ such that $m_0t = m_1 + \sum_{i \geq 2} m_i t_i$ for some $t_i \in J$. Therefore $(m_0n_{kt}) = (\sum_{i \geq 2} m_i x_{n_k}(t_i))$, implying that if we agree to identify 0 and 1\textsuperscript{1} the sequence $\{ (m_0n_{kt}) \}$ converges as $k \to \infty$. This in turn implies that $\{ (n_{kt}) \}_{k \in N}$ has only

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finitely many [in fact, at most \( m_0 = m_0(t) \)] limit points in \( I \) for each \( t \in R \). This is impossible unless \( \{ n_k \}_{k \in N} \) is finite, since otherwise \( \{(n_k t)\}_{k \in N} \) is dense in \( I \) for almost all \( t \in R \) by a theorem of Hardy and Littlewood [1, Theorem 1.40]. That completes the proof.

The index set \( J \) in the above proof may be of measure zero, since \( J \cup \{1\} \) is a Hamel basis of the reals over the rationals, and there exist Hamel bases of measure zero [6]. For this reason the Hardy-Littlewood theorem could not be applied directly to our situation. We could have avoided this slight holdup by initially choosing a nonmeasurable \( J \). Such a \( J \) exists, since there exists a Hamel basis which is not a Lebesgue measurable set [6]. The only measurable Hamel bases have measure zero [6].

References


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