IRREGULAR INVARIANT MEASURES RELATED TO HAAR MEASURE

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Abstract. Let $G$ be a locally compact nondiscrete group, and let $\nu_1$ be a Haar measure on an open subgroup of $G$. It is not hard to show that $\nu_1$ must be the restriction of a Haar measure $\nu$ on all of $G$. Here we show that there exists a translation invariant measure $\mu$ (found by extending $\nu_1$ to the cosets of $H$ in a natural way) which agrees with $\nu$ on, for example, $(\nu)$ $\sigma$-finite sets, open sets, and subsets of $H$. Although $\nu$ can be computed from $\mu$ in a relatively simple manner, the two measures are not equal in general. In fact, there is an extreme case, namely when $H$ is not $\sigma$-compact and has uncountably many cosets, in which $\mu$ fails very badly to be regular—there are closed sets on which $\mu$ is not inner regular and (other) closed sets on which $\mu$ is not outer regular. One condition sufficient for this extreme case to be possible is when $G$ is Abelian and not $\sigma$-compact.

1. Definitions and notation. Let $\mu$ be a (nonnegative, countably additive) measure defined on a $\sigma$-algebra $\mathcal{M}$ of subsets of a topological space $X$. If $S$ is in $\mathcal{M}$, we say that $\mu$ is inner regular on $S$ if

$$\mu S = \sup \{ \mu C : C \subseteq M, C \text{ compact}, C \subseteq S \}.$$ 

We say that $\mu$ is outer regular on $S$ if

$$\mu S = \inf \{ \mu U : U \subseteq M, U \text{ open}, U \supseteq S \}.$$ 

Following [2, 11.34], we say that $\mu$ is regular if it is outer regular on every set in $\mathcal{M}$ and inner regular on every open set in $\mathcal{M}$, and if every compact set in $\mathcal{M}$ has finite measure.

By Haar measure on a locally compact group $G$, we mean a left Haar measure as defined in, e.g., [2]; that is to say, a left-translation invariant, regular, nondegenerate measure on a $\sigma$-algebra $\mathcal{M}(G)$ of subsets of $G$. $\mathcal{M}(G)$ contains all the closed subsets of $G$ and consists of all the sets which are measurable with respect to the Carathéodory outer measure associated with the measure.

If $H$ is a subgroup of $G$ (not necessarily normal), $G/H$ denotes the space of left cosets of $H$ in $G$. If $S$ is a set, $\mathcal{P}(S)$ denotes the collection of all subsets of $S$; $|S|$ denotes the cardinality of $S$. 
2. **Lemma 1.** Let $X$ be a Hausdorff space, $(X, M, \mu)$ a measure space, with $\mu C < \infty$ for all compact $C \in M$ and $\mu$ inner regular on $S \in M$ whenever $\mu S < \infty$. Let

$$\lambda S = \sup \{ \mu C : C \text{ compact}, C \subseteq S \},$$

$$\nu S = \inf \{ \mu U : U \text{ open}, U \in M, U \supseteq S \},$$

for all $S \in M$. Then

1. $\lambda S = \mu S = \nu S$ whenever $\nu S < \infty$,
2. $\lambda$ and $\nu$ are measures.

**Proof.**

(1) Suppose $\nu S < \infty$. Let $V$ be a $G_\delta$ set such that $S \subseteq V$ and $\nu S = \mu V$. If $C$ is a compact member of $M$ with $C \subseteq V - S$, then $S \subseteq V - C \subseteq V$. Now $V - C$ is a $G_\delta$ set so $\mu (V - C) = \nu S = \mu V$, thus $\mu C = 0$. Hence $\mu (V - S) = 0$, and $\mu S = \mu V = \nu S$; $\lambda S = \mu S$ is obvious since $\mu S < \infty$.

(2) Suppose $\{ S_j : j = 1, 2, \ldots \} \subseteq M$ and $S_j \cap S_k = \emptyset$ ($j \neq k$). Let $S = \bigcup S_j$, let $C$ be a $\sigma$-compact subset of $S$ such that $\lambda S = \mu C$ and (for each $j$) let $C_j$ be a $\sigma$-compact subset of $S_j$ with $\mu C_j = \lambda S_j$. Then

$$\lambda S = \mu C = \mu (\bigcup C \cap S_j) = \sum \mu (C \cap S_j) \leq \sum \lambda S_j$$

and

$$\lambda S \geq \mu (\bigcup C_j) = \sum \mu C_j = \sum \lambda S_j,$$

hence $\lambda$ is a measure. Clearly (by (1)),

$$\nu S = \mu S = \sum \mu S_j = \sum \nu S_j$$

if $\nu S < \infty$. If $\nu S = \infty$, take $U_j$ a $G_\delta$ set such that $U_j \supseteq S_j$ and $\nu S_j = \mu U_j$ ($j = 1, 2, \ldots$). Then

$$\sum \nu S_j = \sum \mu U_j \geq \mu (\bigcup U_j) \geq \nu S = \infty,$$

thus $\nu$ is a measure.

**Lemma 2.** Let $G$ be a locally compact group, $H$ an open subgroup of $G$. Then $M(H) = M(G) \cap P(H)$.

**Proof.** Let $\nu$ be a Haar measure on $G$ and $\nu_1$ a Haar measure on $H$. Both $\nu$ and $\nu_1$ are unique to within a multiplicative constant; further, if $U$ is an open subset of $H$ with compact closure, then $0 < \nu U < \infty$ and $0 < \nu_1 U < \infty$. Thus we may assume that $\nu U = \nu_1 U$; but then the Carathéodory outer measures associated with $\nu$ and $\nu_1$, respectively, are equal on $P(H)$. It follows that the $\nu$-measurable and $\nu_1$-measurable subsets of $H$ coincide, which is to say that $M(H) = M(G) \cap P(H)$.
Note. The proof of Lemma 2 contains the information that there is a one-to-one correspondence between the Haar measures on $G$ and $H$, respectively, given by $\nu \leftrightarrow \nu_1 = \nu|M(H)$. One by-product of Theorem 1 will be a method of computing $\nu$, given $\nu_1$.

**Theorem 1.** Let $G$ be a nondiscrete locally compact group, let $H$ be an open subgroup of $G$, and $\nu_1$ a left Haar measure on $H$. For $S \in M(G)$, define

$$
\mu S = \sum \{ \nu_1(xS \cap H) : xH \in G/H \},
$$

$$
\nu S = \inf \{ \mu U : U \text{ open}, U \supset S \}.
$$

Then

1. $\mu$ is a well-defined left-invariant measure on $M(G)$.
2. $\nu$ is a Haar measure for $G$.
3. $\mu$ and $\nu$ are both extensions of $\nu_1$ and $\mu$ and $\nu$ agree on open sets and $(\nu)$ $\sigma$-finite sets.
4. If $H$ is not $\sigma$-compact, $\mu$ fails to be inner regular on some closed subsets of $G$.
5. If $G/H$ is uncountable, $\mu$ fails to be outer regular on some closed subsets of $G$.

**Proof.** (1) We know that if $S \in M(G)$, then $xS \in M(G)$ and therefore $xS \cap H \in M(G) \cap P(H) = M(H)$ for all $x \in G$. Further, if $xH = yH$, then

$$
\nu_1(xS \cap H) = \nu_1(yx^{-1}xS \cap H) = \nu_1(yS \cap H),
$$

since $\nu_1$ is left invariant. Thus $\mu$ is well defined; it is clearly left-invariant. To show that $\mu$ is a measure, suppose $\{ S_j \} \subset M(G)$, $S_j \cap S_k = \emptyset$ ($j \neq k$); then

$$
\mu(\bigcup S_j) = \sum \nu_1(\bigcup xS_j \cap H) = \sum \left( \sum_{j=1}^{\infty} \nu_1(xS_j \cap H) \right)
$$

$$
= \sum_{j=1}^{\infty} \left( \sum_{j} \nu_1(xS_j \cap H) \right) = \sum_{j=1}^{\infty} \mu S_j
$$

(by standard arguments; either both double summations have uncountably many nonzero terms or the $l_1$ version of Fubini's theorem applies).

(2) Let $S \in M(G)$, with $S$ open or $\mu S < \infty$. There is a countable set $\{ x_j \}$ such that $\mu S = \sum_{j=1}^{\infty} \nu_1(x_j S \cap H)$. For each $j$, there is a $\sigma$-compact set $C_j$ such that $C_j \subset x_j S \cap H$ and $\nu_1 C_j = \nu_1(x_j S \cap H)$. Thus

$$
\mu S = \sum \nu_1 C_j = \sum \mu(x_j^{-1} C_j) = \mu(\bigcup x_j^{-1} C_j) \leq \lambda S \leq \mu S,
$$

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where $\lambda$ is as in Lemma 1, since $\bigcup x_j^{-1}C_j$ is a $\sigma$-compact subset of $S$. Now Lemma 1 applies, so that $\nu$ is a regular measure defined on $M(G)$; it is obvious that $\nu$ has the other properties required of a Haar measure.

(3) This statement is obvious.

(4) If $H$ is not $\sigma$-compact, then (16.14) of [2] shows that there is a closed subset of $H$ on which $\nu$ (and therefore $\mu$) is not inner regular.

(5) If $G/H$ is uncountable, then an argument easily derived from the proof of (16.14) (op. cit.) shows that there is a closed subset $F$ of $G$ such that $\mu F = 0$ and $\nu F = \infty$; thus $\mu U = \infty$ for any neighborhood $U$ of $F$ and $\mu$ is not outer regular on $F$.

Notes on Theorem 1. I. Statements (4) and (5) each imply that $G$ is not $\sigma$-compact. Theorems 2 and 3, below, state conditions under which (4) and (5) can be true for the same subgroup $H$.

II. $\lambda$ is the inner-regular “Haar measure” described in [1, Theorem 1] (and, from a different point of view, in [4, II.1])—or, more properly, the extension of this (weakly) Borel measure to $M(G)$.

III. For each $S$ in $M(G)$, one of the following statements must always be true:

(a) $\lambda S = \mu S = \nu S$ ($\mu$ is outer regular and inner regular on $S$).

(b) $\lambda S < \mu S < \nu S = \infty$ ($\mu$ is not outer regular on $S$).

(c) $\lambda S < \mu S < \nu S = \infty$ ($\mu$ is not inner regular on $S$).

IV. If $\nu_1$ were a right Haar measure, one could proceed in the same manner to obtain right-invariant measures on $M(G)$ with the desired properties, except that right cosets of $H$ and right translates of sets would play the rôle given to left cosets and left translates in Theorem 1.

3. Lemma 3. Let $G$ be an uncountable Abelian group. Then $G$ contains a subgroup $K$ such that $|K| = |G/K| = |G|$.

Proof. Let $r = |G|$.

Case 1. Suppose $r = r_0(G)$, the torsion-free rank of $G$. Then there exists a maximal independent torsion-free subset $X$ of $G$ such that $|X| = r$. Let $K = \{x^2 : x \in X\}$. As in [4, II.8], $|K| = |G/K| = r$.

Case 2. Suppose $G$ is torsion. Since $G$ is uncountable, it must have a subgroup $G_1$ of bounded order such that $|G_1| = r$. By (A.25) of [2], $G_1$ is the direct sum of cyclic groups; thus $G_1 = \{Y\}$ where $Y$ is an independent set and $|Y| = r$. Let $Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and $|Y_1| = |Y_2| = r$; let $K = \{Y_1\}$. Then $|K| = r \geq |G/K| \geq |G_1/K| \geq |Y_2| = r$.

Case 3. Suppose $r > r_0(G)$. Let $X$ be a maximal torsion-free independent set; let $G' = G/[X]$. $G'$ is a torsion group and $|G'| = r$, so by
Case 2 \( G' \) has a subgroup \( K' \) such that \( |K'| = |G'/K'| = r \); let \( K \) be the subgroup of \( G \) such that \( K' = K/\langle X \rangle \). Clearly \( |K| = r \), and by the Third Isomorphism Theorem \( |G/K| = |G'/K'| = r \).

**Theorem 2.** Let \( G \) be a locally compact Abelian group which is not \( \sigma \)-compact. Then \( G \) has an open subgroup \( H \) such that \( H \) is not \( \sigma \)-compact and \( G/H \) is uncountable.

**Proof.** Let \( U \) be an open \( \sigma \)-compact subgroup of \( G \); then \( G' = G/U \) is uncountable and by Lemma 3 has a subgroup \( K' \) such that \( |K'| = |G'/K'| = |G'| \). Let \( H \) be the subgroup of \( G \) such that \( K' = H/U \). Then \( H \) is not \( \sigma \)-compact since \( H/U \) is a cover of \( H \) by uncountably many pairwise disjoint open sets. Also, \( |G/H| = |G'/K'| = |G'| \).

**Theorem 3.** Let \( G \) be a locally compact group which is not the union of fewer than \( \aleph_2 \) compact sets. Then \( G \) has an open subgroup \( H \) such that \( H \) is not \( \sigma \)-compact and \( G/H \) is uncountable.

**Proof.** Let \( U \) be a \( \sigma \)-compact open subgroup of \( G \) and let \( H \) be a subgroup of \( G \) generated by a collection of \( \aleph_1 \) cosets of \( U \). \( H \) has the desired properties.

4. **Examples (the group \( R_d \times R \)).** Let \( G \) be the group \( R_d \times R \), where \( R_d \) is the discrete reals and \( R \) is the reals with the usual topology. Let \( \lambda_0 \) be Lebesgue measure on \( R \), and for \( r \in R_d \), let \( \lambda_r(S) = \lambda_0(\{ x: (r, x) \in S \}) \). Define

\[
\lambda S = \sum \{ \lambda_r(S): r \in R_d \}.
\]

**Case 1.** (From [3, §12.58]). Let \( H = \{ 0 \} \times R \) and \( \nu_1 = \lambda \mid M(H) \) = Lebesgue measure on \( \{ 0 \} \times R \). Here \( H \) is \( \sigma \)-compact and \( G/H \) is uncountable, being isomorphic to \( R_d \). We have \( \lambda = \mu \), and \( \mu F_1 = 0 \), \( \nu F_1 = \infty \), where \( F_1 = R_d \times \{ 0 \} \).

**Case 2.** Let \( K \) be the subgroup of \( R_d \) generated by a Hamel basis over \( Q \); let \( H = K \times R \). In this case, \( H \) is not \( \sigma \)-compact and \( G/H \) is uncountable. For \( S \in M(G) \), we have

\[
\nu S = \inf \{ \lambda U: U \text{ open, } U \supseteq S \},
\]

\[
\nu_1 = \nu \mid M(H),
\]

as natural choices for Haar measures. Here,

\[
\mu S = \sum \{ \nu(S \cap (\{ r \} \times R)): r \in K_2 \}
\]

where \( K_2 \) is a subgroup of \( R_d \) such that \( R_d = K_2 \oplus K \). Let \( F_2 = K_2 \times \{ 0 \} \); then \( \lambda F_1 = 0 \), \( \mu F_1 = \nu F_1 = \infty \) and \( \lambda F_2 = \mu F_2 = 0 \), \( \nu F_2 = \infty \).
References


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