SEPARABLE REPRESENTATIONS OF A \( W^* \)-ALGEBRA

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1. Introduction. The decomposition of a \( W^* \)-algebra (i.e. von Neumann algebra) as a direct integral of factors has been done for the case in which the algebra acts on a separable Hilbert space. A tempting possibility for extension of these results to some nonseparable cases would be to assume merely that the dual space \( M^* \) (as a Banach space) of a \( W^* \)-algebra \( M \) is separable in the weak* topology. The purpose of this note is to show that this possibility is vacuous by showing that any \( W^* \)-algebra \( M \) with weak* separable dual space has a weakly closed faithful * representation on a separable Hilbert space. This extends the result of Rosenthal and Arveson which makes the same assertion in the case that \( M \) is abelian [3].

2. Statement and proof of the theorem. Let \( M \) be a \( W^* \)-algebra. We use the abstract approach of Sakai [4] and consider \( M \) as a \( C^* \)-algebra which is the dual of some Banach space \( F \). Then if \( M^* \) is the dual space of \( M \), there is a natural imbedding of \( F \) in \( M^* \). We shall identify \( F \) with its imbedding in \( M^* \) when convenient. It is well known that \( M \) has a weakly closed, faithful, separable representation if and only if \( F \) is norm separable.

We recall some notation from [1, p. 287]. A positive functional \( f \in M^* \) is called normal if \( f \in F \). It is called singular if there is no normal \( g \in M^* \) with \( f \geq g \). If \( f \in M^* \), we may write \( f \) as \( f = f_1 - f_2 + if^3 - if^4 \) where \( f^j \geq 0 \) \((j = 1, 2, 3, 4)\) and the \( f^j \) are uniquely determined by \( ||f_1 - f_2|| = ||f_1|| + ||f_2|| \) and \( ||f^3 - f^4|| = ||f^3|| + ||f^4|| \). We define \( [f] = f_1 + f_2 + f^3 + f^4 \) for any \( f \in M^* \). If \( f \in M^* \), \( f \geq 0 \), we have the unique decomposition \( f = f^* + f^* \), where \( f^* \) is normal and \( f^* \) is singular and both are positive functionals.

**Theorem.** If \( M^* \) is weak* separable, then \( F \) is norm separable (and hence \( M \) has a weakly closed, faithful, separable representation).

**Proof.** Assume that \( \{f_n\}_{n=1}^\infty \) is a countable weak* dense subset of \( M^* \). Let \( f = \sum_{n=1}^\infty (2^n||f_n||)^{-1}[f_n] \), convergence being assured in the norm topology of \( M^* \). Let \( f = f^* + f^* \) be the decomposition of \( f \) into normal and singular parts. Note that \( f \) is a faithful positive functional on \( M \) since \( \{f_n\}_{n=1}^\infty \) is a weak* dense subset of \( M^* \). Thus \( M \) is countably decomposable (i.e., every family of pairwise orthogonal projections is countable), so by [6] there exists an increasing sequence \( \{p_n\} \)

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of projections in $M$ with $V_{n=1}^\infty p_n = 1$ and $f^*(p_n) = 0$ for all $n = 1, 2, \ldots$. Define for each $n = 1, 2, \ldots$ the spaces $F_n = \{p_n g p_n : g \in F\}$ and $M_n = \{p_n a p_n : a \in M\}$ where $p_n g p_n \in M^*$ by $(p_n g p_n)(a) = g(p_n a p_n)$ for $a \in M$. Now $F_n \subset F$ [4] for each $n = 1, 2, \ldots$; so if $F_n$ is separable for each $n = 1, 2, \ldots$, $F$ is also separable. Thus assume that $n$ is a fixed integer for which $F_n$ is not separable.

We remark that $M_n$ is a $W^*$-algebra with predual $F_n$ and dual $p_n M^* p_n$. We shall prove that $p_n f_k p_n \in F_n$ for each $k = 1, 2, \ldots$. Since $\{p_n f_k p_n\}_{k=1}^\infty$ is obviously total over $M_n$, this would prove that $F_n$ is weakly separable and hence norm separable. Now fix $f_k$ and note that $f_k \leq \|f_k\| \cdot 2^k f$ for each $i = 1, 2, 3, 4$. Thus $p_n f_k p_n \leq \|f_k\| 2^k (p_n f_k p_n) = \|f_k\| 2^k (p_n f^* p_n)$, since $f^*(p_n) = 0$ implies $p_n f^* p_n = 0$ by the Schwartz inequality. Thus [5] $p_n f_k p_n$ is normal on $M_n$ ($i = 1, 2, 3, 4$), so $p_n f_k p_n = p_n f_k p_n - p_n f_k p_n + i p_n f_k p_n - i p_n f_k p_n \in F_n$, for any $k = 1, 2, \ldots$. Thus $F_n$ is separable, so $F$ is separable. Q.E.D.

References


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