A NOTE ON CONVEX AND BAZILEVIĆ FUNCTIONS

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1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular, univalent in $|z| < 1$ and map $|z| < 1$ onto a domain which is starlike with respect to the origin. Then we call $f(z)$ a starlike function. It is well known that a function $f(z)$ is starlike in $|z| < 1$ if and only if

$$\Re \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad |z| < 1.$$

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular, univalent in $|z| < 1$ and map $|z| < 1$ onto a convex domain. Then we call $f(z)$ a convex function. It is well known that a regular function $f(z)$ is convex if and only if

$$1 + \Re \frac{zf''(z)}{f'(z)} > 0 \quad \text{in} \quad |z| < 1.$$

Every convex function is a starlike function [4].

Let $L(r)$ denote the length of the closed curve $C(r)$ which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$ and $A(r)$ the area enclosed by $C(r)$.

Recently Thomas [6], [8] has shown that if $f(z)$ is a starlike function, then

$$L(r) \leq 2(\pi A(r))^{1/2} \left(1 + \log \frac{1 + r}{1 - r}\right).$$

(1)

In this note, for the convex functions we obtain a stronger result than (1).

**Theorem 1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a convex function. Then we have

$$L(r) = 0 \left(A(r) \log \frac{1}{1 - r}\right)^{1/2} \quad \text{as} \quad r \to 1.$$

**Proof.** Since $f(z)$ is a univalent function we can get

Received by the editors May 26, 1969.
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\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \leq \int_0^{2\pi} \int_0^r |zf''(z) + f'(z)| \, d\theta \, dp
\]

\[
= \int_0^r \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| \, d\theta \, dp
\]

\[
= \int_0^{r_1} \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| \, d\theta \, dp
\]

\[
+ \int_r^{r_1} \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| \, d\theta \, dp
\]

where \( r_1 \) is a fixed constant and \( 0 < r_1 < r < 1 \).

Therefore we have

\[
L(r) \leq C + \left( \int_0^{r_1} \int_0^{2\pi} \rho |f'(z)|^2 \, d\theta \, dp \right)^{1/2} \left( \int_0^{r_1} \int_0^{2\pi} \frac{1}{\rho} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 \, d\theta \, dp \right)^{1/2}
\]

\[
\leq C + \left( \int_0^{r} \int_0^{2\pi} \rho |f'(z)|^2 \, d\theta \, dp \right)^{1/2} \left( \int_0^{r_1} \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 \, d\theta \, dp \right)^{1/2}
\]

\[
\leq C + \left( A(r) \right)^{1/2} \left( \int_0^{r_1} \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 \, d\theta \, dp \right)^{1/2}
\]

where \( C \) is a bounded constant.

On the other hand, it is well known that (see for instance [3, p. 294])

\[
\int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 \, d\theta \, dp \leq 4\pi \log \frac{1 + r}{1 - r}.
\]

This completes our proof and a question arises whether there is a positive constant \( \alpha \) and a convex function \( f(z) \) for which

\[
L(r) \geq \alpha \left( A(r) \log \frac{1}{1 - r} \right)^{1/2} \quad \text{as } r \to 1.
\]

I can not give an answer for this question.

2. A function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) that is regular in \( |z| < 1 \) is called a Bazilević function of type \( \beta \), if there exists a starlike function \( g(z) \) and \( \beta > 0 \) such that

\[
\text{Re} \frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}} > 0 \quad \text{in } |z| < 1.
\]
Bazilevič [1] has shown that each such function is univalent in \( |z| < 1 \). Every starlike function is a Bazilevič function of type \( \beta \).

Then the following theorems have been obtained in [6], [7] and [2], [5].

**Theorem A.** Let \( f(z) \) be a Bazilevič function of type \( \beta \), \( 0 < \beta \leq 1 \) and let

\[
M(r) = \max_{|z| = r} |f(re^{i\theta})| \leq (1 - r)^{-\alpha} \quad 0 < \alpha \leq 2.
\]

Then

\[
L(r) = O(1 - r)^{-\alpha} \quad \text{as} \quad r \to 1.
\]

**Theorem B.** Let \( f(z) \) be a Bazilevič function of type \( \beta \), \( \arg f(z) \) be a function of bounded variation on \( |z| = r < 1 \) and let

\[
M(r) = \max_{|z| = r} |f(re^{i\theta})| \leq (1 - r)^{-\alpha} \quad 0 < \alpha \leq 2.
\]

Then we have

\[
L(r) = O(1 - r)^{-\alpha} \quad \text{as} \quad r \to 1.
\]

**Theorem 2.** The results of Theorem A and Theorem B are sharp for each \( \alpha, 0 < \alpha \leq 2 \) and therefore \( O \) in (2) and (3) cannot be replaced by \( o \).

**Proof.** It is easily verified that the function

\[
f(z) = \frac{z}{(1 - z)^{\alpha}}, \quad 0 < \alpha \leq 2
\]

is a starlike function [2, p. 216] and \( M(r) \leq (1 - r)^{-\alpha} \).

Applying the theorem of Fejér and Riesz to \( f(z) = z/(1 - z)^{\alpha} \) we have

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta = \int_0^{2\pi} \left| \frac{1 - z + \alpha z}{(1 - z)^{\alpha + 1}} \right| d\theta
\]

\[
\geq 2r \int_{-r}^{r} \frac{1 - \rho + \alpha \rho}{(1 - \rho)^{\alpha + 1}} \, d\rho
\]

\[
= O(1 - r)^{-\alpha} \quad \text{as} \quad r \to 1.
\]

This completes our proof.

The author would like to acknowledge helpful comments made by the referee.
REFERENCES


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