ON DISCONJUGACY CRITERIA

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1. Disconjugacy criteria. In the differential equation,

\[ x^{(n)} + p_0(t)x^{(n-1)} + \cdots + p_{n-2}(t)x' + p_{n-1}(t)x = 0, \]

let the coefficients be real-valued and summable on an interval \( I \).

The equation (1.1) is said to be disconjugate on \( I \) if no solution \( x(t) \neq 0 \) has \( n \) zeros, counting multiplicities, on \( I \). The object of this note is to derive disconjugacy criteria, for a compact interval \( I = [a, b] \) of length \( |I| = b - a \), generally related to conditions of the type

\[ \sum_{k=0}^{n-1} A_k |I|^k \| p_k \| \leq 1, \]

where \( A_0, \ldots, A_{n-1} \) are constants and \( \| \cdots \| \) is the \( L^1(I) \) norm. For \( n = 2 \), a known disconjugacy condition is

\[ 2^{-1} \| p_0 \|_1 + 2^{-2} |I| \cdot \| p_1^+ \| \leq 1, \]

where \( r^+ = \max(0, r) \) and \( r^- = \max(0, -r) \), and follows from Levin's condition \([5]\),

\[ (\exp \frac{1}{2} \| p_0 \|_1) \| p_1^+ \| \leq 4/ |I|, \]

by virtue of \( e^{-x} \geq 1 - x \) for \( x \geq 0 \); cf. also \([0]\). If \( p_0 = 0 \), then (1.3) and (1.4) reduce to a condition of Lyapunov; e.g., \([1, p. 346]\).

Introduce the constants

\[ C_n = C(n - 1, \lfloor n/2 \rfloor) / n!, \]

where \( \lfloor r \rfloor \) is the largest integer not exceeding \( r \) and \( C(n, j) = n! / j!(n - j)! \) is the binomial coefficient. These numbers \( C_1 = 1, C_2 = 1/2, C_4 = 1/3, \)

\( C_4 = 1/8, \ldots \) appear in Levin \([3]\), \([4]\) and Hukuhara \([2]\). Hukuhara \([2]\) gives a disconjugacy criterion related to (1.2) with

\[ A_0 = 1 \quad \text{and} \quad A_k = 2^{-k}C_k \quad \text{for} \quad k = 1, \ldots, n - 1. \]

Using his method, this will be sharpened to (1.12) below.

In \([6]\), Nehari states that (1.2) is a disconjugacy criterion if

Received by the editors January 28, 1969.

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(1.7) \[ A_k = 2^{-k-1} \quad \text{for} \quad k = 0, \ldots, n - 1. \]

It has been noted by Nehari (see [0]) that his proof of this assertion is incorrect and that the validity of the assertion is undecided. It will be seen that Nehari’s statement is correct and, in fact, is contained in (1.9) below. For, on the one hand, a simple calculation shows that \((C_k/4)[k/(k+1)]^{k-1} \leq 2^{-k-1}\) for \(k = 1, 2, 3\) and, on the other hand, it is easy to verify that \(C_k/4 < 2^{-k-1}\) if \(k \geq 4\). (In [7], Zaiceva asserts a refinement of criterion (1.7), but her inequality (16) on p. 765 does not seem correct, and this invalidates her proof.)

I would like to thank Professor A. M. Fink for calling my attention to the question of the validity of the criterion (1.7) and for a helpful correspondence in the course of the preparation of this note.

**Theorem 1.** The following are disconjugacy criteria for (1.1) on \(I = [a, b]\):

(i) \(p_0, \ldots, p_{n-1}\) satisfy

\[
(1.8) \quad (\exp \frac{1}{2} \|p_0\|) \sum_{k=1}^{n-1} (C_k/4)[k/(k+1)]^{k-1} \|I\|^k \|p_k\| \leq 1,
\]

in particular,

\[
(1.9) \quad \frac{1}{2} \|p_0\| + \sum_{k=1}^{n-1} (C_k/4)[k/(k+1)]^{k-1} \|I\|^k \|p_k\| \leq 1;
\]

(ii) if, in addition, \(p_{m}(t) = \ldots = p_{m-1}(t) = 0\) for some \(m, 2 \leq m \leq n - 1\), then (1.8) can be relaxed to

\[
(1.10) \quad \exp(\frac{1}{2} \|p_0\|) \sum_{k=m}^{n-1} (C_k/2^{m+1})[k/(k+m)]^{k-m} \|I\|^k \|p_k\| \leq 1;
\]

(iii) \(p_0, \ldots, p_{n-1}\) satisfy

\[
(1.11) \quad \left(\exp \|p_0\|\right) \left\{ \sum_{k=1}^{n-1} [(C_k/2^{k+1}) \|I\|^k \|p_k\|]^2 \right. \\
+ \frac{1}{2} \sum_{j \neq k} \sum_{j+k} C_j C_k j^{j+k}(j+k)^{-j-k} \|I\|^{j+k} \|p_j\| \|p_k\| \right\} \leq 1;
\]

(iv) the quantities \(Q_1, Q_2\) satisfy

\[
(1.12) \quad \max(Q_1, Q_2) \leq 1,
\]

where

\[
(1.13) \quad Q_1 = \left(\exp \int_a^{(a+b)/2} p_0^+ dt\right) \sum_{k=1}^{n-1} C_k 2^{-k} \|I\|^k \|p_k\| dt,
\]
(1.14) \[ Q_2 = \left( \exp \int_{(a+b)/2}^{b} p_0^a dt \right) \sum_{k=1}^{n-1} C_k 2^{-k} I | k \int_{(a+b)/2}^{b} | p_k | dt. \]

2. The Levin inequalities. Theorem 1 will be deduced from the following inequalities given by Levin [4] and Hukuhara [3]. (The latter also makes the assumption \( a \leq a_0 \leq a_1. \))

**Lemma 2.1.** Let \( n \geq 1, \ I = [a, \ b], \) and \( x(t) \in C^n(I). \) Let \( x(t), x'(t), \ldots, x^{(n-1)}(t) \) vanish at points of \( I, \) say

\[ x^{(k)}(a_k) = 0 \quad \text{for } k = 0, \ldots, n - 1, \]

and let

\[ a \leq a_1 \leq \cdots \leq a_{n-1} \leq b. \]

Then

\[ \max_I | x(t) | \leq MC_n | I |, \quad \text{where } M = \max_I | x^{(n)} | \]

and \( C_n \) is defined by (1.5). \( C_n \) cannot be replaced by a smaller constant.

Levin's proof [4] uses some functional analysis and the Krein-Milman theorem on extreme points of convex sets. Hukuhara's proof [2] is an elementary calculus proof. We shall give a different, short, elementary proof which will yield sharper inequalities (cf. (2.6)-(2.7), (2.8)-(2.9), (3.5) and (4.3) below) and has the following consequence:

**Corollary 2.1.** The assertion (2.3) in Lemma 2.1 can be strengthened to

\[ \int_{a}^{b} | x'(t) | dt \leq MC_n | I |, \]

whether or not \( x(t) \) has a zero \( t = a_0 \) on \( I. \)

**Remark.** It is easy to verify that equality in (2.3) holds if \( | x^{(n)}(t) | \equiv M, \ a_k = a_{n-1} \) for \( k \geq n/2, \) \( a_k = a_0 \) for \( k < n/2, \) and \( a = a_0, \ b = a_{n-1}. \) Furthermore, the proof of Lemma 2.2 will imply that equality cannot hold in any other cases; cf. (2.6a)–(2.7a) and (3.5) below.

**Lemma 2.2.** Strengthen (2.2) in Lemma 2.1 to

\[ a \leq a_0 \leq \cdots \leq a_{n-1} \leq b \]

and let \( a_n = b. \)

Then, on \([a, a_0],\)

\[ | x(t) | \leq \int_{t}^{b} | x'(s) | ds \leq \int_{t}^{b} | x^{(n)}(s) | (s - t)^{n-1} ds/(n - 1); \]
and, on $[a_k, a_{k+1}]$, for $k = 0, \cdots, n-1$,

$$(n - 2)! \left| x(t) \right| \leq (n - 2)! \int_{a_0}^t \left| x'(s) \right| \, ds$$

(2.7nk)

$$\leq \sum_{j=0}^{k-1} C(n - 2, j) \int_{a_j}^{a_{j+1}} (t - r)^j F_j(r) \, dr$$

$$+ C(n - 2, k) \int_{a_k}^t (t - r)^k F_k(r) \, dr,$$

where

$$F_j(r) = \int_0^r \left| x^{(n)}(s) \right| (s - r)^{n-2} \, ds.$$ 

In particular, if $M = \max \left| x^{(n)} \right|$ on $[a, b]$, then

(2.8) $n! \left| x(t) \right| \leq M [(b - t)^n - (b - a_0)^n]$ on $[a, a_0]$,

(2.9) $n! \left| x(t) \right| \leq M [(b - a_0)^n - (b - t)^n] \max_{0 \leq j \leq k} C(n - 1, j)$ on $[a_k, a_{k+1}]$ for $k = 0, 1, \cdots, n-1$.

Although the inequalities (2.6n), (2.7nk) may not be in a very useful form, the point of these relations is that they have a simple proof and imply (2.8), (2.9) and Lemma 2.1.

3. Proof of Lemma 2.2. On (2.6n). The relation (2.61) is trivial on $[a, a_0]$. Assume $n \geq 2$ and (2.6n-1). Thus (2.6n-1) applied to $x'(t)$ gives

(3.1) $\left| x'(t) \right| \leq \int_t^b \left| x^{(n)}(s) \right| (s - t)^{n-2} ds/(n - 2)!$ on $[a, a_1]$.

For $a \leq t \leq a_0$, the inequality

$\left| x(t) \right| \leq \int_t^{a_0} \left| x'(r) \right| \, dr \leq \int_t^b \left| x'(r) \right| \, dr$

combined with (3.1) gives (2.6n).

On (2.7nk). The inequality (3.1) and $\left| x(t) \right| \leq \int_{a_0}^t \left| x'(r) \right| \, dr$ for $t \geq a_0$ gives (2.7n0) for $n = 2, 3, \cdots$. Assume (2.7nk) for fixed $k \geq 0$ and all $n \geq k + 2$. We shall verify (2.7n,k+1). From (2.7nk) for $t = a_{k+1}$,

$$(n - 2)! \int_{a_0}^{a_{k+1}} \left| x'(s) \right| \, ds$$

(3.2)

$$\leq \sum_{j=0}^k C(n - 2, j) \int_{a_j}^{a_{j+1}} (a_{k+1} - r)^j F_j(r) \, dr;$$

and (2.7n-1,k) applied to $x'$ gives
\[(n - 3)! \mid x'(v) \mid \leq \sum_{j=0}^{k-1} C(n - 3, j) \int_{a_{j+1}}^{a_{j+2}} (v - r)^{j} F_{j+1}(r) dr
\]
\[(3.3) \quad + C(n - 3, k) \int_{a_{k+1}}^{v} (v - r)^{k} F_{k+1}(r) dr \]
for \(a_{k+1} \leq v \leq a_{k+2}\) and \(k = 0, \ldots, n - 2\). A quadrature of (3.3),
\[(n - 2)! \int_{a_{k+1}}^{t} \mid x'(v) \mid dv
\]
\[(3.4) \quad \leq \sum_{j=1}^{k} C(n - 2, j) \int_{a_{j}}^{a_{j+1}} [(t - r)^{j} - (a_{k+1} - r)^{j}] F_{j}(r) dr
\]
\[+ C(n - 2, k + 1) \int_{a_{k+1}}^{t} (a_{k+1} - r)^{k+1} F_{k+1}(r) dr, \]
together with (3.2) gives (2.7\textsubscript{n,k+1}).

On (2.8). By (3.1), \(\mid x'(t) \mid \leq M(b - t)^{n-1}/(n - 1)! \) on \([a, a_{0}]\). A quadrature over \([t, a_{0}]\) gives (2.8).

On (2.9). The right side of (2.7\textsubscript{nk}) is not decreased if \(\mid x^{(n)}(s) \mid \) is replaced by \(M\) and, in the integrand, \(t\) is replaced by \(b\). Thus, on \([a_{k}, a_{k+1}]\),
\[n! \mid x(t) \mid \leq M \sum_{j=0}^{k-1} C(n - 1, j) [(b - a_{j})^{n} - (b - a_{j+1})^{n}] \]
\[(3.5) \quad + MC(n - 1, k) [(b - a_{k})^{n} - (b - t)^{n}]. \]
This implies (2.9) and completes the proof of Lemma 2.2.

4. Proof of Corollary 2.1. The proof of (2.7\textsubscript{nk}) shows that if \(a \leq a_{0} \leq a_{1}\) and \(a_{k} \leq t \leq a_{k+1}\), then
\[n! \int_{a_{0}}^{t} \mid x'(s) \mid ds \leq M \sum_{j=0}^{k-1} C(n - 1, j) [(b - a_{j})^{n} - (b - a_{j+1})^{n}] \]
\[(4.3) \quad + MC(n - 1, k) [(b - a_{k})^{n} - (b - t)^{n}], \]
whether or not \(x(a_{0}) = 0\). This inequality, with \(a = a_{0}, k = n - 1,\) and \(t = b = a_{n}\), implies (2.4), since \(n! C_{n} = \max C(n - 1, j)\) for \(0 \leq j \leq n - 1\).

5. Proof of Theorem 1. Suppose that (1.1) has a solution \(x(t) \neq 0\) with \(n\) zeros, counting multiplicities, on \([a, b]\). It will be shown that the inequalities (1.8), (1.10), (1.11) and (1.12) cannot hold.
We can suppose that there are numbers
\[a = a_{0} \leq a_{1} \leq \cdots \leq a_{n-1} = b_{n-1} \leq b_{n-2} \leq \cdots \leq b_{1} \leq b_{0} = b,\]
such that \( x^{(k)}(t) = 0 \) if \( t = a_k \) and \( t = b_k \). Put \( c = a_{n-1} = b_{n-1} \), \( a < c < b \). The analogues of Lemma 2.1 are applicable to each of the intervals \([a, c] \), \([c, b]\). Actually, the analogues will be applied to \( x^{(n-1-k)} \), rather than to \( x \) with \( n-1 \) replaced by \( k \), \( k = 1, \ldots, n-2 \). Let 
\[ M = \max |x^{(n-1)}(t)| \text{ on } [a, c] \text{ and let } |x^{(n-1)}(r)| = M, a < r < c. \]

Write (1.1) as
\[
\left( x^{(n-1)} \exp \int_a^t p_0 ds \right)' + \left( \exp \int_a^t p_0 ds \right) \sum_{k=1}^{n-1} p_k(t) x^{(n-1-k)} = 0.
\]

An integration over the interval \([r, c]\) gives
\[
M \exp \int_a^r p_0 ds < \sum_{k=1}^{n-1} \max |x^{(n-1-k)}| \int_r^c \left( \exp \int_a^t p_0 ds \right) |p_k| dt.
\]

By Lemma 2.1 and
\[
\exp \int_r^t p_0 ds \leq \exp \int_a^c p_0 ds \text{ for } r \leq t \leq c,
\]
we get
\[
(5.1) \quad 1 < \left( \exp \int_a^c p_0^+ ds \right) \sum_{k=1}^{n-1} C_k(c - a)^k \int_a^c |p_k| dt.
\]

Similarly,
\[
(5.2) \quad 1 < \left( \exp \int_c^b p_0^- ds \right) \sum_{k=1}^{n-1} C_k(b - c)^k \int_c^b |p_k| dt.
\]

On (1.12). Since either \( c \leq (a+b)/2 \) or \( c \geq (a+b)/2 \), that is, \( c-a \leq (b-a)/2 \) or \( b-c \leq (b-a)/2 \), (5.1) and (5.2) imply that either \( Q_1 > 1 \) or \( Q_2 > 1 \).

On (1.8). By (5.1) and (5.2), we have
\[
1 < \left( \exp \|p_0\| \right) \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} C_j C_k (c - a)^k (b - c)^j
\]
\[
(5.3) \quad \cdot \int_a^c |p_k| dt \int_c^b |p_j| dt.
\]

Note that
\[
(5.4) \quad (c - a)^k (b - c)^j \leq \frac{1}{(k+j)(k+j)^+} \quad \text{for } a \leq c \leq b
\]

If \( m \geq 0 \), then \( jk/(j+k)^2 \leq 1/4 \) implies that...
If \( i \leq m = l \), then

\[
(5.6) \quad j^k/(j + k)^{i+k} \leq 2^{-2m} [j/(j + k)]^{j-m} [k/(j + k)]^{k-m}.
\]

The inequalities (5.3), (5.4) and (5.6), with \( i = m = 1 \), show that

\[
1 < (\exp \| f_0 \|) \sum_{k=1}^{n-1} (C_k/2)[k/(k + 1)]^{k-1} | I |^{k} \int_{a}^{b} | p_k | \, dt
\]

\[
\times \left\{ \sum_{j=1}^{n-1} (C_j/2)[j/(j + 1)]^{j-1} | I |^{j} \int_{c}^{d} | p_j | \, dt \right\}.
\]

Take the square root of both sides of this inequality and use the arithmetic-geometric mean inequality, applied to the last two factors, to obtain

\[
1 < (\exp \frac{1}{2} \| f_0 \|) \sum_{k=1}^{n-1} (C_k/4)[k/(k + 1)]^{k-1} | I |^{k} \int_{a}^{b} | p_k | \, dt.
\]

On (1.10). This is proved in the same way using \( i = m \) in (5.6).

On (1.11). Insert the inequality (5.4) into (5.3), interchange \( j \) and \( k \), and add the resulting inequalities to obtain

\[
2 < (\exp \| f_0 \|) \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} C_j C_k j^{j} k^{k}(j + k)^{-j-k} | I |^{j+k}
\]

\[
\cdot \left\{ \int_{a}^{c} | p_k | \, dt \int_{c}^{b} | p_j | \, dt + \int_{c}^{b} | p_k | \, dt \int_{a}^{c} | p_j | \, dt \right\}.
\]

The factors \( \left\{ \cdots \right\} \) satisfy

\[
\left\{ \cdots \right\} \leq \| p_j \| \cdot \| p_k \| \quad \text{for all} \quad j, k;
\]

but if \( j = k \),

\[
\left\{ \cdots \right\} = 2 \int_{a}^{c} | p_k | \, dt \int_{c}^{b} | p_k | \, dt \leq \frac{1}{2} (\| p_k \|)^2 \quad \text{for} \quad j = k.
\]

The last three formula lines contradict (1.11) and complete the proof of Theorem 1.

References


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