SOME REGULARITY CONDITIONS FOR THE (f, d_n, z_1) SUMMABILITY METHOD

A. JAKIMOVSKI AND H. SKERRY

Abstract. Regularity conditions for the generalized Lototsky summability method (f, d_n, z_1) are derived which do not restrict the Taylor coefficients of f. Some examples are given.

1. Introduction. In [4] Lototsky defined a summability method of some importance which was subsequently generalized by Jakimovski in [3] to his (F, d_n) method, and this in turn was generalized by Smith in [5] to the method (f, d_n). Finally, Smith generalized (f, d_n) to (f, d_n, z_1) in [6]. All of the sufficiency conditions for regularity given by Smith require that the power series expansion of f about the origin have real nonnegative coefficients. In this paper we give some sufficiency conditions which do not so require.

2. Definitions and preliminaries.

Definition 2.1. Let f be a nonconstant function holomorphic at the origin and let \{d_n\}_{n=1}^\infty be a sequence of complex numbers with f(z_1)+d_n \neq 0, where z_1 is in the disk on which f is holomorphic. Let

\[
\prod_{k=1}^n (f(z)+d_k) = \sum_{k=0}^\infty \rho_{nk} z^k, \quad n \geq 1.
\]

Then the method (f, d_n, z_1) is defined by the matrix (a_{nk}), where

\[
a_{0k} = 1, \quad k = 0
\]

\[
= 0, \quad k > 0 \quad \text{and} \quad a_{nk} = \frac{\rho_{nk} z_1^k}{\prod_{i=1}^n (f(z_i)+d_i)}, \quad n \geq 1.
\]

In terms of the above definition the Lototsky, (F, d_n), and (f, d_n) methods are, respectively, the methods (z, n-1, 1), (z, d_n, 1), and (f, d_n, 1). It is easy to see that if z_1 \neq 0 and g(z)=f(z_1z), then the methods (f, d_n, z_1) and (g, d_n) are the same.

It is well known that a matrix A=(a_{nk}) is regular if and only if the following conditions are met:

\[
\sup_n \sum_{k=0}^\infty |a_{nk}| < \infty,
\]
(2.3) \[ \lim_{n} \sum_{k=0}^{\infty} a_{nk} = 1, \]
(2.4) \[ \lim_{n} a_{nk} = 0, \quad k \geq 0. \]

Definition 2.1 insures that (2.3) is always satisfied, so we need only concern ourselves with the remaining criteria.

For convenience we will use the notation \( \prod_{1}^{n} (b + d_{k}) = (b + d_{n})!, \)
and we will set \( d_{n} = \rho_{e}^{\theta_{n}}, \quad -\pi < \theta_{n} \leq \pi. \)

3. Regularity conditions for \((f, d_{n}, z_{i})\).

**Theorem 3.1.** Suppose (3.2) \( f \) is holomorphic on the disk \( |z| < \lambda, \)
\( 0 < |z_{1}| = R < \lambda, \) (3.3) \( |f(z_{1})| = r \) and \( |f(0)| = \rho < r, \) and (3.4) \( \{f(z)\}_{n}^{\infty} = \sum_{k=0}^{\infty} c_{nk}z^{k} \) \( R^{k} \leq M r^{n}, n \geq 1. \) Then \((f, d_{n}, z_{i})\) is regular
if \((F, d_{n}/f(z_{1}))\) is regular.

**Proof.** Let \( (z + d_{n})! = \sum_{k=0}^{n} b_{nk}z^{k}. \) The entries in the \((F, d_{n}/f(z_{1}))\) matrix are the coefficients of the polynomial
(3.5) \[ \prod_{1}^{n} (z + d_{i}/f(z_{1})) = \frac{(f(z_{1})z + d_{n})!}{(f(z_{1}) + d_{n})!} = \sum_{k=0}^{n} \frac{b_{nkf(z_{1})}^{k}}{f(z_{1}) + d_{n}!} z^{k}. \]

Now,
\[ (f(z) + d_{n})! = \sum_{j=0}^{n} b_{njf(z_{1})}^{j} = \sum_{j=0}^{n} b_{nj} \sum_{k=0}^{\infty} c_{jk} z^{k} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n} b_{nj} c_{jk} \right) z^{k}, \]
so from Definition 2.1 follows \( p_{nk} = \sum_{j=0}^{n} b_{nj} c_{jk}. \) Then,
\[ \sum_{k=0}^{\infty} \left| a_{nk} \right| = \sum_{k=0}^{\infty} \left| \frac{z_{1}}{(f(z_{1}) + d_{n})!} \sum_{j=0}^{n} b_{nj} c_{jk} \right| \]
\[ \leq \frac{1}{|f(z_{1}) + d_{n}|!} \sum_{k=0}^{\infty} \sum_{j=0}^{n} \left| b_{nj} c_{jk} z_{1}^{k} \right| \]
\[ = \frac{1}{|f(z_{1}) + d_{n}|!} \sum_{j=0}^{n} \left| b_{nj} \sum_{k=0}^{\infty} \left| c_{jk} \right| R^{k} \leq M \sum_{j=0}^{n} \left| \frac{b_{njf(z_{1})}^{j}}{(f(z_{1}) + d_{n})!} \right| \right. \]

The regularity of \((F, d_{n}/f(z_{1}))\) and (3.5) imply \( \sum_{k=0}^{\infty} \left| a_{nk} \right| < H, n \geq 0. \)
From the formula
\[ a_{nk} = \sum_{j=0}^{n} \frac{b_{njf(z_{1})}^{j}}{(f(z_{1}) + d_{n})!} \frac{z_{1}^{k} c_{jk}}{f(z_{1})^{j}} \]

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it is clear that for a given \( k \), \( a_{nk} \) is the \( nth \) term of the \( (F, d_n/f(z_1)) \) transform of the sequence \( \{ z_i c_{jk}/f(z_1) \}_{j=0}^\infty \). In view of the assumed regularity of \( (F, d_n/f(z_1)) \), to demonstrate (2.4) we need only show this to be a null sequence for each \( k \). For \( k=0 \) we have

\[
\left| c_{j0}/f(z_1)^j \right| = (\rho/r)^j \to 0, \quad j \to \infty.
\]

For fixed \( k > 0 \), Newton's multinomial expansion yields

\[
c_{jk} = \sum_{i_0 + \cdots + i_m = j, \ i_1 + 2i_2 + \cdots + mi_m = k} \frac{j!}{i_0!i_1! \cdots i_m!} c_{i_0} \cdots c_{i_m},
\]

Clearly, the number of \( m \)-tuples \( (i_1, \ldots, i_m) \) is no greater than \( k^j \), and \( i_0 \leq j \), so the number of terms in the above sum is at most \( k^{j}j \). If \( 1 \leq v \leq m \), then \( i_v \leq k \), so \( \left| c_{i_1} \cdots c_{i_m} \right| \leq B \) for all choices of the exponents and of \( m \leq k \). Also,

\[
\frac{j!}{i_0!i_1! \cdots i_m!} \leq \frac{j!}{i_0!} \leq \frac{j!}{(j-k)!} \leq j^k, \quad j \geq k,
\]

and \( \left| c_{i_0} \right| = \rho^{i_0} = \rho^j/\rho^j \leq K \rho^j \) since \( 0 \leq j - i_0 \leq k \). Thus for \( j \geq k \) we have the estimate

\[
\left| c_{jk}/f(z_1)^j \right| \leq k^j j BK \rho^j/r^j \to 0 \quad \text{as} \ j \to \infty.
\]

We will need the following three lemmas.

**Lemma 3.5.** If \( z = \rho e^{i\theta} \), then

\[
0 \leq \frac{1 + \rho}{|1 + z|} - 1 \leq \frac{2\rho(1 - \cos \theta)}{|1 + z|^2}.
\]

**Proof.**

\[
0 \leq \frac{1 + \rho}{|1 + z|} - 1 \leq \left( \frac{1 + \rho}{|1 + z|} \right)^2 - 1 = (1 + \rho)^2 - (1 + 2\rho \cos \theta + \rho^2) \leq \frac{2\rho(1 - \cos \theta)}{|1 + z|^2}.
\]

**Lemma 3.6.** Suppose \( d_k = \rho_k e^{i\theta_k} \) and

\[
\sum_{1}^{\infty} \frac{\rho_k(1 - \cos \theta_k)}{|1 + d_k|^2} < \infty.
\]
Then

$$1 \leq \prod_1^\infty \frac{1 + \rho_k}{1 + d_k} < \infty.$$ 

**Proof.** This follows immediately from the above lemma.

**Lemma 3.7.** Suppose at most finitely many $d_k$'s are 0,

$$\prod_1^\infty \frac{1 + \rho_k}{1 + d_k} < \infty,$$

and

$$\sum_1^\infty \frac{1}{1 + d_k} = \infty.$$

Then

$$\lim_{n \to \infty} \prod_1^n \frac{d_k}{1 + d_k} = 0,$$

the prime indicating that the product is over all nonzero factors.

**Proof.** First, observe that

$$\sum_1^\infty \left( \frac{1 + \rho_k}{1 + d_k} - 1 \right) < \infty.$$ 

Now, let the set of positive integers be the union of the sets

$$P = \{ n \mid 1 - \rho_n/|1 + d_n| > 0 \} \text{ and } Q = \{ \text{all other positive integers} \}.$$ 

If $k \in Q$, then

$$\frac{1}{1 + d_k} \leq \frac{1 + \rho_k}{1 + d_k} - 1,$$

whence

$$\sum_{k \in Q} |1 + d_k|^{-1} < \infty.$$ 

It follows that $\sum_{k \in P} |1 + d_k|^{-1} = \infty$, and thus

$$\sum_{k \in P} \left( \frac{1}{|1 + d_k|} - \left( \frac{1 + \rho_k}{|1 + d_k|} - 1 \right) \right)$$

$$= \sum_{k \in P} \left( 1 - \frac{\rho_k}{|1 + d_k|} \right) = \infty,$$

which in turn implies that the product

$$(3.9) \quad \prod_{k \in P} \frac{\rho_k}{1 + d_k} \text{ diverges to 0, i.e., } \lim_{n \to \infty} \prod_{k \leq n, k \in P} \frac{d_k}{1 + d_k} = 0.$$
From (3.8) it is clear that
\[
\sum_{k \in \mathbb{Q}} \left( \frac{\rho_k}{|1 + d_k|} - 1 \right) = \sum_{k \in \mathbb{Q}} \left\{ \left( \frac{1 + \rho_k}{|1 + d_k|} - 1 \right) - \frac{1}{|1 + d_k|} \right\} < \infty,
\]
so
\[
(3.10) \quad \prod_{k \in \mathbb{Q}} \frac{\rho_k}{|1 + d_k|}
\]
converges. The conclusion follows easily from (3.9) and (3.10).

Lemma 3.2 and formula (3.12) of [3] together imply that if only finitely many \(d_n = 0\),
\[
(3.11) \quad \frac{(1 + \rho_n)!}{|1 + d_n|!} \leq K, \quad n = 1, 2, \ldots ,
\]
and \((d_n/(1+d_n))'! \to 0\) as \(n \to \infty\), then \((F, d_n)\) is regular. Lemma 3.3 and (3.12) of [3] imply that if infinitely many \(d_n = 0\), then (3.11) alone suffices for the regularity of \((F, d_n)\). Thus it follows from Lemma 3.7 that (3.11) and \(\sum_1^\infty |1+d_n|^{-1} = \infty\) give the regularity of \((F, d_n)\).

From Theorem 3.1 we now get

**Theorem 3.12.** Let (3.2), (3.3), and (3.4) hold and suppose
\[
\frac{(r + \rho_n)!}{|f(z_1) + d_n|!} \leq K, \quad n \geq 1,
\]
and
\[
\sum_1^\infty |f(z_1) + d_n|^{-1} = \infty .
\]

Then \((f, d_n, z_1)\) is regular.

**Corollary 3.13.** Let \(f(z_1) = r e^{i\phi}\) and let (3.2), (3.3), and (3.4) hold. Suppose \(\sum_1^\infty |f(z_1) + d_n|^{-1} = \infty\) and
\[
\sum_1^\infty \frac{(\text{Im}(d_n e^{-i\phi})^{1/2})^2}{|f(z_1) + d_n|} = \frac{1}{2} \sum_1^\infty \frac{\rho_n[1 - \cos(\theta_n - \phi)]}{|f(z_1) + d_n|^2} < \infty .
\]

Then \((f, d_n, z_1)\) is regular.

**Proof.** By Lemma 3.6,
\[
\sum_{1}^{\infty} \frac{\rho_n[1 - \cos(\theta_n - \phi)]}{|f(z_1) + d_n|^2} < \infty \text{ implies } \sum_{1}^{\infty} \frac{1 + \rho_n/r}{|1 + d_n/f(z_1)|} = \prod_{1}^{\infty} \frac{r + \rho_n}{|f(z_1) + d_n|} < \infty,
\]
so Theorem 3.12 gives the result.

**Corollary 3.14.** Let the Taylor expansion of \( f \) about the origin have real nonnegative coefficients and radius of convergence \( \lambda > 1 \). Suppose

\[
\sum_{1}^{\infty} \frac{1}{|f(1) + d_n|} = \infty \quad \text{and} \quad \sum_{1}^{\infty} \left( \frac{\Im \sqrt{d_n}}{|f(1) + d_n|} \right)^2 < \infty.
\]

Then \((f, d_n)\) is regular.

This is Theorem 2.3 in [5].

**4. Examples.** We can easily generate examples with nonreal Taylor coefficients. Let \( \beta > 0 \) and define \( f(z) = \psi/(1 + i\beta z) = \psi \sum_{0}^{\infty} (-i\beta)^k z^k \), \( \psi \neq 0 \) arbitrary. Then

\[
\frac{f(z)}{1 + i\beta z} = \sum_{k=0}^{\infty} \frac{\psi^n}{r^k} (n + k - 1) (-i\beta)^k z^k = \sum_{k=0}^{\infty} c_{nk} z^k.
\]

If we let

\[
F(z) = \frac{1}{(1 - z)^n} = \sum_{k=0}^{\infty} \frac{(n + k - 1)}{r^k} z^k,
\]
then \( \sum_{k=0}^{\infty} R^k |c_{nk}| = |\psi|^n F(R\beta) \), provided \( R\beta < 1 \), and this is \( O(r^n) \) if \( |\psi|/r \leq 1 - R\beta \). Since \( f \) has radius of convergence \( 1/\beta \), we may choose \( z_1 = i/2\beta \) and \( d_n = n\psi \) to satisfy all the requirements.

Condition (3.4) is probably the hardest to satisfy when constructing examples. In this connection, the following theorem of Bajšanski [1] and Clunie and Vermes [2] gives conditions under which \( \sum_{k=0}^{\infty} |c_{nk}| = O(1) \); it is then relatively easy to meet the remaining restrictions.

**Theorem 4.1.** Let \( f \) be holomorphic on the closed unit disc and let \(|f(z)| < 1\) for \(|z| = 1\) except at finitely many points \( \xi \) at which \(|f(\xi)| = 1\). Then, if \( \{f(z)\}^n = \sum_{k=0}^{\infty} c_{nk} z^k \), it follows that \( \sum_{k=0}^{\infty} |c_{nk}| = O(1) \) if and only if \( \text{Re } A_\xi \neq 0 \) for each such \( \xi \), where \( A_\xi = [i(z - 1)]^n \) is the lead term of the Taylor expansion about 1 of \( h_\xi(z) - z^\alpha_\xi \), and where \( h_\xi(z) = f(\xi z)/f(\xi) \) and \( \alpha_\xi = h_\xi'(1) \).

In [2] it is shown that the functions below satisfy the conditions of Theorem 4.1 and have \( f(1) = 1 \).
\begin{align}
(4.2) \quad f(z) &= \frac{1}{2 + i} (1 + iz + z^2), \\
(4.3) \quad f(z) &= \frac{\omega z - 1}{\omega - 1} \exp\{\omega(z - 1)\}, \quad \omega = e^{i\pi/3}, \\
(4.4) \quad f(z) &= \exp\{\eta(z - 1) - \eta^2(z^2 - 1)\}, \quad \eta = e^{i\theta}, \cos \theta = 1/4.
\end{align}

References


Tel-Aviv University, Michigan State University and Lehigh University