SHORTER NOTE

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AN INEQUALITY FOR RATIONAL FUNCTIONS

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ABSTRACT. An inequality of A. A. Gončar concerning the relative sizes of rational functions on different sets is reinterpreted. This allows a very simple proof of a more general inequality.

A. A. Gončar [1] proved the following: Let G be a doubly connected domain in the Riemann sphere bounded by continua E_1 and E_2. Then if r is any rational function of degree (number of poles) n, one has

$$\min_{z\in E_2} |r(z)| \leqq \rho^n \max_{z\in E_1} |r(z)| \tag{1}$$

where ρ is the modulus of the domain G. (This means that G is conformally equivalent to the annulus $1<|z|<\rho$.)

In this note we present a simple proof of the generalization of this inequality to arbitrary disjoint closed sets E_1 and E_2 of positive (logarithmic) capacity.

Let $g(z, \zeta)$ denote Green's function for the complement of E_1 and define $1/\log \rho$ to be the capacity of E_2 relative to this kernel. Thus $1/\log \rho$ is the maximum of $\mu(E_2)$ for all nonnegative Borel measures μ on E_2 satisfying

$$\int g(z, \zeta) d\mu(z) \leqq 1, \qquad \zeta \notin E_1. \tag{2}$$

It is a fact, not difficult to prove, that ρ is unchanged if the roles of E_1 and E_2 are interchanged. In case E_1 and E_2 bound a doubly connected domain G then ρ as just defined is the modulus of G [2, pp. 96–97].

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We shall show that (1) holds in the general case. We may assume that the maximum appearing on the right side of (1) is 1. If r has poles at $\zeta_1, \cdots, \zeta_n$ then

$$\log|r(z)| - \sum_{k=1}^{n} g(z, \zeta_k)$$

is subharmonic in the complement of E_1 and has lim sup at most 0 at each point of E_1. Therefore by the maximum principle for subharmonic functions

$$\log|r(z)| \leqq \sum_{k=1}^{n} g(z, \zeta_k) \qquad z \notin E_1.$$

Now let μ be any measure on E_2 satisfying (2). Then

$$\mu(E_2) \min_{z \in E_2} \log|r(z)| \leqq \sum_{k=1}^{n} \int g(z, \zeta_k) d\mu(z) \leqq n,$$

and (1) follows immediately from the definition of ρ.

It can be shown [3] that the constant ρ appearing on the right side of (1) is best possible.

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