HOMOTOPY TORSION IN CODIMENSION TWO KNOTS

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1. Introduction. The homology theory of the infinite cyclic covering space of a codimension two knot complement is reasonably well known, but less is known concerning the homotopy theory of the knot complement itself. If \( k: S^n \to S^{n+2} \) is a smooth embedding, let \( S = S^{n+2} - k(S^n) \) denote the open knot complement. One of the module structures on the homotopy of \( S \) is the natural structure over the ring \( \Gamma = \text{integral group ring of } \pi_1(S) \). We say that \( \pi_i(S) \ (i \geq 2) \) has \( \Gamma \)-torsion if there exists \( 0 \neq c \in \pi_i(S), 0 \neq w \in \Gamma \) such that \( 0 = w \cdot c \in \pi_i(S) \).

This paper gives a technique for constructing slice (null-cobordant) knots with \( \Gamma \)-torsion. The homotopy torsion in these examples is related to the homology torsion in \( \tilde{S} \), the infinite cyclic cover of \( S \). If \( \Lambda \) denotes the integral group ring of the infinite cyclic group (generated by \( t \)), then \( H_i(\tilde{S}; Z) \) is a finitely-generated \( \Lambda \)-module for all \( i \).

In the construction, the \( \Lambda \)-generator of \( H_i(\tilde{S}; Z) \neq 0 \) is an embedded \( S^i \). Furthermore the \( \Lambda \)-torsion in \( H_i(\tilde{S}; Z) \) arises from an embedded bounded punctured \( D^{i+1} \). This punctured \( D^{i+1} \) has as boundary a finite disjoint union of copies of \( S^i \), and piping them together inside \( D^{i+1} \) yields an embedded sphere which bounds an embedded disc. This gives a nontrivial \( \Gamma \)-relation in \( \pi_i(S) \).

If \( h_1: \pi_1(S) \to H_1(S; Z) \) is the Hurewicz homomorphism, then \( h_1 \) extends uniquely to the ring homomorphism \( h_1: \Gamma \to \Lambda \). \( \Lambda \) is thought of as the ring of Laurent polynomials in a variable \( t \) with integer coefficients. If \( 0 \neq \lambda \in \Lambda \), then \( \Lambda / \Lambda' \) denotes the cokernel of the injection

\[
\Lambda \to \Lambda',
\]

(see Levine [5]).

We prove the following

**Theorem 1.** Given any polynomial \( F(t) \in \Lambda \) such that \( F(1) = \pm 1 \), then there exists a smooth slice knot \( (S^4, kS^2) \) such that

(i) \( H_1(\tilde{S}; Z) \cong \Lambda / F(t), H_2(\tilde{S}; Z) \cong \Lambda / F(t^{-1}) \);

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(ii) \( \pi_2(S) \) has \( \Gamma \)-torsion \( w \cdot c = 0 \) where \( 0 \neq c \in \pi_2(S) \) and \( 0 \neq w \in \Gamma \);

(iii) \( h_1(w) = F(t^{-1}) \), and \( h_2: \pi_2(\tilde{S}) \to H_2(\tilde{S}; \mathbb{Z}) \) the Hurewicz homomorphism then \( h_2(c) \) is the \( \Lambda \)-generator of \( H_2(\tilde{S}; \mathbb{Z}) \).

One of the knots produced in Theorem 1 is a counterexample to a theorem announced by C. H. Giffen.

The techniques developed in the proof of Theorem 1 can be used to prove the following

**Theorem 2.** Given any polynomial \( F(t) \in \Lambda \), \( F(1) = \pm 1 \), and integers \( n, p \) such that \( n \geq 2 \), \( 1 \leq p < (n+1)/2 \) then there exists a smooth slice knot \( (S^{n+2}, k) \) such that

(i) \( H_p(\tilde{S}; \mathbb{Z}) \cong \Lambda / F(t), \ H_{n+1-p}(\tilde{S}; \mathbb{Z}) \cong \Lambda / F(t^{-1}), \ H_r(\tilde{S}; \mathbb{Z}) = 0 \) if \( r \neq 0, p, n+1-p \); 

(ii) \( \pi_{n+1-p}(S) \) has \( \Gamma \)-torsion \( w \cdot c = 0 \), \( 0 \neq c \in \pi_{n+1-p}(S) \), \( 0 \neq w \in \Gamma \); 

(iii) \( h_1(w) = F(t^{-1}) \), and \( h_{n+1-p}: \pi_{n+1-p}(\tilde{S}) \to H_{n+1-p}(\tilde{S}; \mathbb{Z}) \) the Hurewicz homomorphism then \( h_{n+1-p}(c) \) is the \( \Lambda \)-generator of \( H_{n+1-p}(\tilde{S}; \mathbb{Z}) \); 

(iv) if \( p \geq 2 \) then 

\[
\pi_i(S) \cong \pi_i(S^1), \quad i \leq p - 1,
\]

and

\[
\pi_p(S) \cong \pi_p(\tilde{S}) \xrightarrow{h_p} H_p(\tilde{S}; \mathbb{Z}).
\]

Note that Theorem 2 avoids the case \( n = 2q+1 \) (\( q \geq 2 \)) and \( p = q \). In this case it turns out that

\[
\pi_i(S) \cong \pi_i(S^1), \quad i \leq q - 1,
\]

\[
\pi_q(S) \cong \pi_q(\tilde{S}) \xrightarrow{h_q} H_q(\tilde{S}; \mathbb{Z}) \cong \Lambda / F(t)F(t^{-1}).
\]

This arises because the construction relies on a certain duality between dimensions \( p \) and \( n+1-p \), and in the above mentioned case, dimension \( q \) is selfdual.

We can prove a similar result for ball pairs. Let \( k': B^{n+2} \to B^{n+3} \) denote proper smooth embedding of balls, and \( B = B^{n+3} - k'(B^{n+1}) \) denote the open knot complement, and \( \tilde{B} \) the infinite cyclic cover of \( B \).
Theorem 3. Given $F(t) \in \Lambda$, and integers $n, p$ such that $2 \leq n, 2 \leq p \leq n$ then there exists a smooth knotted ball pair $(B^{n+3}, k'B^{n+1})$ such that

(i) $H_p(\tilde{B}; \mathbb{Z}) \cong A / F(t)$, $H_r(\tilde{B}; \mathbb{Z}) = 0$, $r \neq 0, p$;

(ii) $\pi_p(B) \cong A, \pi_p(\tilde{B}) \cong H_p(\tilde{B}; \mathbb{Z}), \pi_i(B) \cong \pi_i(S^1), i \leq p - 1$.

II. The construction. We will begin with a special case of the proof of Theorem 1. Suppose that $F(t) = (2 - t)$. The knot is constructed in the same manner as those in [3], [7], [8], [9]. Let $(B^5, B^3)$ denote the unknotted ball pair, and $B^* = B^5 - B^3$ the open complement. Let $f: S^0 \to \partial B^*$ be an embedding, and attach a 1-handle to $B^*$ by $f$, rounding corners to obtain the smooth manifold $K = B^* \cup_f h^1$ [Figure 1].

Let $\cong$ denote diffeomorphism, $\simeq$ denote homotopy equivalence, and $\vee$ denote wedge product. Then
As in Figure 1, let $\alpha$ denote the generator of $\pi_1(\partial K)$ which goes around the handle and $\beta$ the generator which links once the "flat" $S^2$ in $\partial K$. Let $g: S^1 \to \partial K$ be the smooth embedding given in Figure 2. $g(S^1)$ is an embedding in the homotopy class of $\alpha^2 \beta \alpha^{-1} \beta^{-1} \in \pi_1(\partial K)$. If $\ast$ is the base point of $S^3$, then in the product structure of $\partial K \approx S^1 \times S^3 - S^2$, $S^1 \times \ast$ is an embedding in the homotopy class of $\alpha$, and we will henceforth identify $\alpha$ with $S^1 \times \ast$. Now from Figure 2, $g(S^1)$ is clearly diffeotopic to $\alpha$ in $\partial K \cup S^2$. Since $\alpha$ has a trivial normal bundle in $\partial K$ then so does $g(S^1)$. Add a 2-handle to $K$ by $g$, rounding corners to obtain $B = K \cup_{\alpha} h^2 = B \ast \cup_{f} h^1 \cup_{g} h^2$. Now $B \cup B^3 = B^* \cup B^3 \cup_{f} h^1 \cup_{g} h^2 \approx B^8 \cup_{f} h^1 \cup_{a} h^2 \approx B^8$. The last diffeomorphism is from the handle cancellation theorem, since the attaching sphere of the 2-handle intersects the belt sphere of the 1-handle transversely at one point. Therefore $B$ is a knot complement; that is $B = B^8 - k'B^3$ for some smooth proper embedding $k': B^3 \to B^3$. Furthermore $\partial B = S = S^4 - kS^2$ is the complement of a smooth slice knot, where $k = k'| \partial B^3$.

**III. Homotopy calculations.** We have $B \approx S^1 \cup S^1 \cup D^2$, and $\pi_1(B)$
\( = (a, \beta | \alpha^2 \beta \alpha^{-1} \beta^{-1}) \). By considering the trace of the surgery associated with the addition of the 2-handle to \( K \), we see that the inclusion \( i: S \to B \) induces the isomorphism

\[ i_\#: \pi_1(S) \cong \pi_1(B). \]

That is, let \( W \) be the trace (Figure 3), such that \( \partial^- W = \partial K \) and \( \partial^+ W = S \). Now if \( D^2 \) represents the core of \( h^2 \) and \( D^3 \) represents the transverse disc of \( h^2 \), then \( \partial K \cup D^2 \simeq W \simeq S \cup D^2 \). So \( \pi_1(S) \cong \pi_1(\partial K \cup D^2) \).

By considering the trace of the surgery associated with the addition of the 1-handle to \( B^* \), we can see that \( \pi_1(\partial K \cup D^3) \cong \pi_1(K \cup D^3) = \pi_1(B) \).

We will now consider the structure of \( \pi_2(S) \) as a module over \( \Gamma \). Let \( b \) be the base point of \( S^1 \). In the product structure of \( \partial K \) let \( S^2_b = b \times S^2 \) be the belt sphere of \( h^1 \). [See Figure 2.] Suppose that \( x_0, x_1, x_2 \in S^1 \) are 3 distinct points such that \( S^2_b \cap g(S^1) = g(x_0) \cup g(x_1) \cup g(x_2) \). Let \( G: S^1 \times D^3 \to \partial K \) be a trivialization of the tubular neighborhood of \( g(S^1) \) in \( \partial K \) such that \( G(S^1 \times D^3) \cap S^2_b = G(x_0 \times D^3) \cup G(x_1 \times D^3) \cup G(x_2 \times D^3) \).

Now \( \partial(S^2_b \cap S) = d_0 \cup d_1 \cup d_2 \) where \( d_i = \partial G(x_i \times D^3), \ i = 0, 1, 2 \) are em-
bedded 2-spheres. That is, \((S^3_b \cap S)\) is a punctured \(D^3\). Figure 4 shows the situation in \(\partial K\) near \(S^3_b\). The hatched area is \(S^3_b \cap S\). With choice of paths to the base point \(*\) as given in Figure 4, \(d_i \in \pi_2(S)\). Furthermore, \(d_1 = \alpha^{-1} \cdot d_0\) where \(\cdot\) denotes \(\pi_1(S)\)-action on \(\pi_2(S)\). This can be seen by moving \(d_0\) around the boundary of the tubular neighborhood of \(g(S^1)\) until it coincides (except for the path to \(*\)) with \(d_1\), and then comparing the resulting paths to the base point under \(\pi_1(S)\)-action. Similarly, \(d_2 = \beta^{-1} \cdot d_1 = \beta^{-1} \cdot (\alpha^{-1} \cdot d_0)\). Now changing the orientation of \(d_2\), we can pipe the boundary components of \(S^3_b \cap S\) together inside \(S^3_b \cap S\) (Figure 5), obtaining an embedding in the homotopy class of \(d_0 + d_1 - d_2 = (1 + \alpha^{-1} - \beta^{-1} \alpha^{-1}) \cdot d_0 \in \pi_2(S)\). Clearly this sphere bounds an embedded \(D^3\), so \((1 + \alpha^{-1} - \beta^{-1} \alpha^{-1}) \cdot d_0 = 0 \in \pi_2(S)\). As will be shown later, \(d_0 \neq 0\) in \(\pi_2(S)\) because \(d_0\), a lift of \(d_0\) in the infinite cyclic cover \(\tilde{S}\), generates \(H_2(\tilde{S}'; \mathbb{Z}) \neq 0\) as a \(\Lambda\)-module. Furthermore \(h_1(1 + \alpha^{-1} - \beta^{-1} \alpha^{-1}) = 2 - t^{-1} \neq 0\) in \(\Lambda\), so \((1 + \alpha^{-1} - \beta^{-1} \alpha^{-1}) \neq 0\) in \(\Gamma\). Hence \(\pi_2(S)\) has \(\Gamma\)-torsion.

More generally, if \(F(t) \in \Lambda\) and \(F(t) = \sum_{t=0}^{m} a_t t^i\) \((a_0 > 0)\) then the construction proceeds exactly as before, with the exception that the 2-handle is attached by the element
\[ \alpha^a \beta \alpha^a \beta \alpha^a \beta \cdots \beta \alpha^m \beta^{-m} \in \pi_1(\partial K). \]

(This is the formula produced in [7].) In this case the element \( w \in \Gamma \) producing the torsion is

\[ w = \sum_{k=0}^{a_0-1} \alpha^{-k} + \text{sgn} \ a_1 \left[ \left( \sum_{k=0}^{a_1-1} \alpha^{-k} \right) \beta^{-1} \alpha^{a_0-1} \right] + \cdots \]

\[ + \text{sgn} \ a_m \left[ \left( \sum_{k=0}^{1|a_m|-1} \alpha^{-k} \right) \beta^{-1} \alpha \beta^{-1} \cdots \beta^{-1} \alpha^{a_0-1} \right] \]

with the convention that if \( a_q = 0 \) then \( \text{sgn} \ a_q = 0 \) and \( \alpha^{1|a_q|-1} = 1 \).

**IV. The counterexample.** Giffen [2, pp. 191] has made the following statement (restated for the smooth case):

"Let \( S = S^4 - k(S^2) \) be a smooth knot complement. Then \( \pi_2(S) \) is the free Abelian group generated by the symbols \( a_e \), where \( 1 \neq e \in [\pi_1(S), \pi_1(S)] \) the commutator subgroup of \( \pi_1(S) \). The action of \( \pi_1(S) \) on \( \pi_2(S) \) is that induced by basis permutation.
Consider the knot produced in detail in the special case of Theorem 1. If the above statement is correct, then $d_0 \in \pi_2(S)$ has an expansion as a finite linear combination of nontrivial elements of $[\pi_1(S), \pi_1(S)]$ (dropping $a_c$ and keeping $c$), that is $d_0 = \sum_{i=1}^{q} m_i \xi_i$ where $\{m_i\}$ are integers and $\xi_i \in [\pi_1(S), \pi_1(S)]$. Further $\xi_i \neq \xi_j$ for $i \neq j$, and $m_i \neq 0$ for all $i$. It turns out in this case that $[\pi_1(S), \pi_1(S)]$ is Abelian and $\alpha^{-1} \in [\pi_1(S), \pi_1(S)]$. This will be proved in the next section. Allowing this, we compute the torsion relation in terms of the basis and the action: $\alpha^{-1} \in [\pi_1(S), \pi_1(S)]$ and action by conjugation means that the action of $\alpha^{-1}$ on $\pi_2(S)$ is trivial, so

$$(1 + \alpha^{-1} - \beta^{-1} \alpha^{-1}) \cdot d_0 = (2 - \beta^{-1}) \cdot d_0 = 0.$$ 

Or

$$(1) \quad 2 \sum_{i=1}^{q} m_i \xi_i - \sum_{i=1}^{q} m_i \beta^{-1} \xi_i \beta = 0.$$ 

Since the action of $\pi_1(S)$ is by permutation on the basis elements, (1) can hold iff the action of $\beta^{-1}$ on $\{\xi_i\}_{i=1}^{q}$ is permutation within the set. So we can take it instead to be a permutation $\rho$ on the indexing set of the coefficients—that is if $\beta^{-1} \xi_i \beta = \xi_j$ then $m_i \beta^{-1} \xi_i \beta$ becomes $m_{\rho(i)} \xi_j$. So (1) is true iff the coefficients in each coordinate are zero; or $2m_i - m_{\rho(i)} = 0$ for all $i$. But, as a permutation on $q$ objects, $\rho$ has finite order, say $r \geq 1$. Then $m_1 = (1/2)m_{\rho(1)} = (1/4)m_{\rho^2(1)} = \cdots = (1/2)^r m_{\rho^r(1)} = (1/2)^r m_1$. Since $m_1 \neq 0$ then $1 = (1/2)^r$, a contradiction.

What seems to be wrong with Giffen’s statement is the description of the action of $\pi_1(S)$ on $\pi_2(S)$. For a correct description of the structure of $\pi_2(S)$ as a $\Gamma$-module in the case of fibered knots, see [1].

V. The infinite cyclic cover $\bar{S}$. In order to complete the proof of Theorem 1, we will construct the infinite cyclic covering space $\bar{S}$ of $S$, and study its homotopy and homology. We will produce a handlebody decomposition for $\bar{B}$ (the infinite cyclic cover of $B$), and the induced covering of $S$ will be $\bar{S}$. The unknotted open complement $B^* = B^3 - B^3$ fibers over $S^1$ with fiber $\hat{D}^4$, a half-closed $D^4$. That is, $\hat{D}^4$ has a closed $D^3$ (corresponding to the unknotted $B^3$) removed from its boundary. So the infinite cyclic cover of $B^*$ is $\bar{B}^* = R^1 \times \hat{D}^4$. The attaching sphere $f(S^0)$ of $h^1$ lifts to a family of embedded spheres $f_i: S^0 \to \partial \bar{B}^*$, where $i$ is associated with $t \in J(t)$—the infinite cyclic group of covering transformations of $\bar{B}^*$. Add the family of 1-handles $\{h^1\}$ to $\bar{B}^*$ by $f_i$ to get $\bar{K}$, an infinite cyclic cover of $K$. Repeat the
process for the 2-handles to get $\tilde{B}$. Figure 6 shows $\partial \tilde{K}$ with the family $\{g_i\}$ of attaching spheres for the $\{h^2_i\}$. Let $\tilde{S}_0 = \partial \tilde{K}$, and $\tilde{S} = \partial \tilde{B}$ be the induced infinite cyclic cover of $S$. All integral homology groups in this situation are finitely-generated $\Lambda$-modules, and the exact sequences which follow are taken to be exact sequences of $\Lambda$-modules. The calculations follow those in [5], [8].

\[ H_i(\tilde{S}_0) = \Lambda \quad i = 1, 3, \]
\[ = 0 \quad \text{otherwise.} \]
Let $G_i: S^1 \times D^3 \rightarrow \tilde{S}_0$ be a trivialization of the tubular neighborhood of the attaching sphere $g_i(S^1)$ of $h^2$, where $G_i$ is a lift of $G: S^1 \times D^3 \rightarrow \partial K$. Let $\tilde{S}_i = \tilde{S}_0 - \bigcup_i G_i(S^1 \times D^3)$ where upper bar denotes topological closure. By excision

$$H_i(\tilde{S}_0, \tilde{S}_i) = \Lambda \quad i = 3, 4,$$

$$= 0 \quad \text{otherwise}.$$  

Likewise

$$H_i(\tilde{S}, \tilde{S}_i) = \Lambda \quad i = 2, 4,$$

$$= 0 \quad \text{otherwise}.$$  

Let $\Lambda(\tilde{a})$ denote the free $\Lambda$-module generated by the symbol $\tilde{a}$. Let $\tilde{a}_0$ denote the lift of $a$ at level $t^0$ in $\tilde{S}$ [Figure 6]. Let $\tilde{\gamma}_0$ be the lift of $\tilde{S}^3$ at level $t^0$ in $\tilde{S}$. Let $c_0 = G(x_0 \times D^3)$, $\tilde{c}_0 = G_0(x_0 \times D^3)$ and $\tilde{d}_0 = \partial \tilde{c}_0 = G_0(x_0 \times S^2)$, the lift of $d_0$. With this choice of generators we have $H_1(\tilde{S}_0) = \Lambda(\tilde{a}_0)$, $H_3(\tilde{S}_0) = \Lambda(\tilde{\gamma}_0)$, $H_2(\tilde{S}_0, \tilde{S}_1) = \Lambda(\tilde{c}_0)$, $H_2(\tilde{S}, \tilde{S}_1) = \Lambda(\tilde{e}_0)$ where $\tilde{e}_0$ is the lift at level $t^0$ of the image of the core of $h^2$ after it has been pushed out onto the boundary of the handle.

Consider the exact sequence for the pair $(\tilde{S}_0, \tilde{S}_1)$:

$$\begin{array}{c}
H_3(\tilde{S}_0) & \rightarrow & H_3(\tilde{S}_0, \tilde{S}_2) & \rightarrow & H_2(\tilde{S}_1) & \rightarrow & 0 \\
\Lambda(\tilde{\gamma}_0) & \rightarrow & \Lambda(\tilde{c}_0)
\end{array}$$

where the homomorphism $\cdot \{g_i\}$ is as follows:

$$\tilde{\gamma}_0 \cdot \{g_i\} = \left[ \sum_{t \in \{t\}} t \langle \tilde{\gamma}_0 \cdot g_i \rangle \right] \tilde{c}_0$$

where $\langle \tilde{\gamma}_0 \cdot g_i \rangle$ is the intersection number of $\tilde{\gamma}_0$ and $g_i(S^1)$. (See [4, p. 515].) In this case $\cdot \{g_i\}(\tilde{\gamma}_0) = F(t^{-1})\tilde{c}_0$. So $\cdot \{g_i\}$ is a monomorphism, and $H_2(\tilde{S}_1)$ is presented as a $\Lambda$-module by the $1 \times 1$ matrix $F(t^{-1})$. Moreover, the $\Lambda$-generator of $H_2(\tilde{S}_1)$ is $\partial \tilde{e}_0 = \tilde{d}_0$.

Consider now the exact sequence for the pair $(\tilde{S}, \tilde{S}_1)$:

$$\begin{array}{c}
0 & \rightarrow & H_2(\tilde{S}_1) & \rightarrow & H_2(\tilde{S}) & \rightarrow & H_2(\tilde{S}, \tilde{S}_1) & \rightarrow & H_1(\tilde{S}_0) \\
\Lambda(\tilde{e}_0) & \rightarrow & \Lambda(\tilde{c}_0) & \rightarrow & \Lambda(\tilde{d}_0)
\end{array}$$
Since $\partial \tilde{a}_0 = F(t)\tilde{a}_0$, then $\partial$ is a monomorphism and $H_2(\tilde{S}) \cong H_2(\tilde{S})$. Therefore $d_0 \neq 0$ in $H_2(\tilde{S})$; and hence $d_0 \neq 0$ in $\pi_2(S)$. This completes the proof of Theorem 1.

Since $\tilde{S}$ is the infinite cyclic cover of $S$, then $\pi_1(\tilde{S}) \cong [\pi_1(S), \pi_1(S)]$. From Figure 6 for $F(t) = (2-t)$, we see that $\pi_1(\tilde{S})$ is isomorphic to the group on infinitely many generators $\{\tilde{a}_i\}$, with infinitely many relations $\{\tilde{a}_i^2 = 1\}$, which is Abelian because the generators commute. Furthermore, with correct choice of base point in $\tilde{S}$, $\tilde{a}_0$ is the lift of $a \in \pi_1(S)$, hence $a^{-1} \in [\pi_1(S), \pi_1(S)]$. This completes the counter-example.

Theorem 2 is proved by construction in much the same way as Theorem 1, and is in fact easier to do when $p > 1$. If $p = 1$, the construction is exactly the same as in Theorem 1, only it takes place on the unknotted ball pair $(B^{n+3}, B^{n+1})$. If $p > 1$, then attach $h^p$ to $B^* = B^{n+3} - B^{n+1}$ by a nullisotopic $S^{p-1}$ in $\partial B^*$, obtaining $K = B^* \cup h^p$. As before

$$K \approx S^p \times D^{n+3-p} - B^{n+1} \quad K \approx S^1 \cup S^p$$

$$\partial K \approx S^p \times S^{n+2-p} - S^n \quad \partial K \approx S^1 \cup S^p \cup S^{n+2-p}.$$ 

Attach $h^{p+1}$ to $K$ by an embedding in the homotopy class of $F(t) \cdot a$, where $a$ generates $\pi_p(\partial K)$. The knot produced has all the characteristics listed in Theorem 2.

Theorem 3 is proved in exactly the same way as Theorem 2. The sphere pairs constructed in Theorem 2 bound ball pairs. These ball pairs have the characteristics listed in Theorem 3. However, the hypothesis of Theorem 3 allows $(n+1)/2 \leq p \leq n$. For $p$ in this range (in and above the middle dimension of the boundary sphere pair), the construction can still be done. (See [8].)

Theorem 3 provides a generalization of the results of T. M. Price [6]. Using Zeeman's twist-spinning, he produces a $(B^6, k'B^4)$ with boundary $(S^8, kS^8)$ such that $\pi_1(B^6 - k'(B^4)) = Z$ but $\pi_1(S^8 - k(S^8)) \neq Z$. Theorem 3 improves this example by one dimension, i.e. $(S^4, kS^2) = \partial(B^5, k'B^3)$ such that $\pi_1(B^5 - k'(B^3)) = Z$, $\pi_1(S^4 - k(S^2)) \neq Z$. Furthermore, $\pi_1(S^4 - k(S^2))$ and $\pi_2(B^5 - k'(B^3))$ can be specified as in Theorems 2 and 3.

**References**


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