HOMOTOPY TORSION IN CODIMENSION TWO KNOTS[1]

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1. **Introduction.** The homology theory of the infinite cyclic covering space of a codimension two knot complement is reasonably well known, but less is known concerning the homotopy theory of the knot complement itself. If $k: S^n \to S^{n+2}$ is a smooth embedding, let $S = S^{n+2} - k(S^n)$ denote the open knot complement. One of the module structures on the homotopy of S is the natural structure over the ring Γ = integral group ring of $\pi_1(S)$. We say that $\pi_i(S)$ $(i \geqq 2)$ has Γ-torsion if there exists $0 \neq c \in \pi_i(S)$, $0 \neq w \in \Gamma$ such that $0 = w \cdot c \in \pi_i(S)$.

This paper gives a technique for constructing slice (null-cobordant) knots with Γ-torsion. The homotopy torsion in these examples is related to the homology torsion in $\tilde{S}$, the infinite cyclic cover of S. If Λ denotes the integral group ring of the infinite cyclic group (generated by t), then $H_i(\tilde{S}; Z)$ is a finitely-generated Λ-module for all i. In the construction, the Λ-generator of $H_i(\tilde{S}; Z) \neq 0$ is an embedded S^i. Furthermore the Λ-torsion in $H_i(\tilde{S}; Z)$ arises from an embedded bounded punctured D^{i+1}. This punctured D^{i+1} has as boundary a finite disjoint union of copies of S^i, and piping them together inside D^{i+1} yields an embedded sphere which bounds an embedded disc. This gives a nontrivial Γ-relation in $\pi_i(S)$.

If $h_1: \pi_1(S) \to H_1(S; Z)$ is the Hurewicz homomorphism, then h_1 extends uniquely to the ring homomorphism $\bar{h}_1: \Gamma \to \Lambda$. Λ is thought of as the ring of Laurent polynomials in a variable t with integer coefficients. If $0 \neq \lambda \in \Lambda$, then Λ/λ denotes the cokernel of the injection

$$\Lambda \xrightarrow{\lambda} \Lambda$$
$$1 \mapsto \lambda$$

(see Levine [5]).

We prove the following

THEOREM 1. *Given any polynomial $F(t) \in \Lambda$ such that $F(1) = \pm 1$, then there exists a smooth slice knot (S^4, kS^2) such that*

(i) $H_1(\tilde{S}; Z) \cong_\Lambda \Lambda/F(t)$, $H_2(\tilde{S}; Z) \cong_\Lambda \Lambda/F(t^{-1})$;

Presented to the Society, January 24, 1968; received by the editors January 27, 1969, and, in revised form, August 8, 1969.

[1] This research supported by NSF Grant GP-5458.

(ii) *$\pi_2(S)$ has Γ-torsion $w\cdot c=0$ where $0\neq c\in\pi_2(S)$ and $0\neq w\in\Gamma$;*

(iii) *$\bar{h}_1(w)=F(t^{-1})$, and $h_2\colon\pi_2(\tilde{S})\to H_2(\tilde{S};Z)$ the Hurewicz homomorphism then $h_2(c)$ is the Λ-generator of $H_2(\tilde{S};Z)$.*

One of the knots produced in Theorem 1 is a counterexample to a theorem announced by C. H. Giffen.

The techniques developed in the proof of Theorem 1 can be used to prove the following

THEOREM 2. *Given any polynomial $F(t)\in\Lambda$, $F(1)=\pm 1$, and integers n, p such that $n\geqq 2$, $1\leqq p<(n+1)/2$ then there exists a smooth slice knot (S^{n+2}, kS^n) such that*

(i) *$H_p(\tilde{S};Z)\cong_\Lambda \Lambda/F(t)$, $H_{n+1-p}(\tilde{S};Z)\cong_\Lambda \Lambda/F(t^{-1})$, $H_r(\tilde{S};Z)=0$ $r\neq 0$, p, $n+1-p$;*

(ii) *$\pi_{n+1-p}(S)$ has Γ-torsion $w\cdot c=0$, $0\neq c\in\pi_{n+1-p}(S)$, $0\neq w\in\Gamma$;*

(iii) *$\bar{h}_1(w)=F(t^{-1})$, and $h_{n+1-p}\colon\pi_{n+1-p}(\tilde{S})\to H_{n+1-p}(\tilde{S};Z)$ the Hurewicz homomorphism then $h_{n+1-p}(c)$ is the Λ-generator of $H_{n+1-p}(\tilde{S};Z)$;*

(iv) *if $p\geqq 2$ then*

$$\Gamma \underset{\cong}{\overset{\bar{h}_1}{\longrightarrow}} \Lambda,$$

$$\pi_i(S)\cong\pi_i(S^1),\qquad i\leqq p-1,$$

and

$$\pi_p(S)\underset{\Lambda}{\cong}\pi_p(\tilde{S})\underset{\cong_\Lambda}{\overset{h_p}{\longrightarrow}} H_p(\tilde{S};Z).$$

Note that Theorem 2 avoids the case $n=2q+1$ $(q\geqq 2)$ and $p=q$. In this case it turns out that

$$\pi_i(S)\cong\pi_i(S^1),\qquad i\leqq q-1,$$

$$\pi_q(S)\underset{\Lambda}{\cong}\pi_q(\tilde{S})\underset{\cong_\Lambda}{\overset{h_q}{\longrightarrow}} H_q(\tilde{S};Z)\underset{\Lambda}{\cong}\Lambda/F(t)F(t^{-1}).$$

This arises because the construction relies on a certain duality between dimensions p and $n+1-p$, and in the above mentioned case, dimension q is selfdual.

We can prove a similar result for ball pairs. Let $k'\colon B^{n+2}\to B^{n+3}$ denote proper smooth embedding of balls, and $B=B^{n+3}-k'(B^{n+1})$ denote the open knot complement, and $\tilde{B}$ the infinite cyclic cover of B.

THEOREM 3. *Given $F(t) \in \Lambda$, and integers n, p such that $2 \leqq n$, $2 \leqq p \leqq n$ then there exists a smooth knotted ball pair $(B^{n+3}, k'B^{n+1})$ such that*

(i) $H_p(\tilde{B}; Z) \cong_\Lambda \Lambda/F(t)$, $H_r(\tilde{B}; Z) = 0$, $r \neq 0, p$;

(ii) $\pi_p(B) \cong_\Lambda \pi_p(\tilde{B}) \underset{\cong\Lambda}{\xrightarrow{h_p}} H_p(\tilde{B}; Z)$, $\pi_i(B) \cong \pi_i(S^1)$, $i \leqq p-1$.

II. **The construction.** We will begin with a special case of the proof of Theorem 1. Suppose that $F(t) = (2-t)$. The knot is constructed in the same manner as those in [3], [7], [8], [9]. Let (B^5, B^3) denote the unknotted ball pair, and $B^* = B^5 - B^3$ the open complement. Let $f: S^0 \to \partial B^*$ be an embedding, and attach a 1-handle to B^* by f, rounding corners to obtain the smooth manifold $K = B^* \cup_f h^1$ [Figure 1].

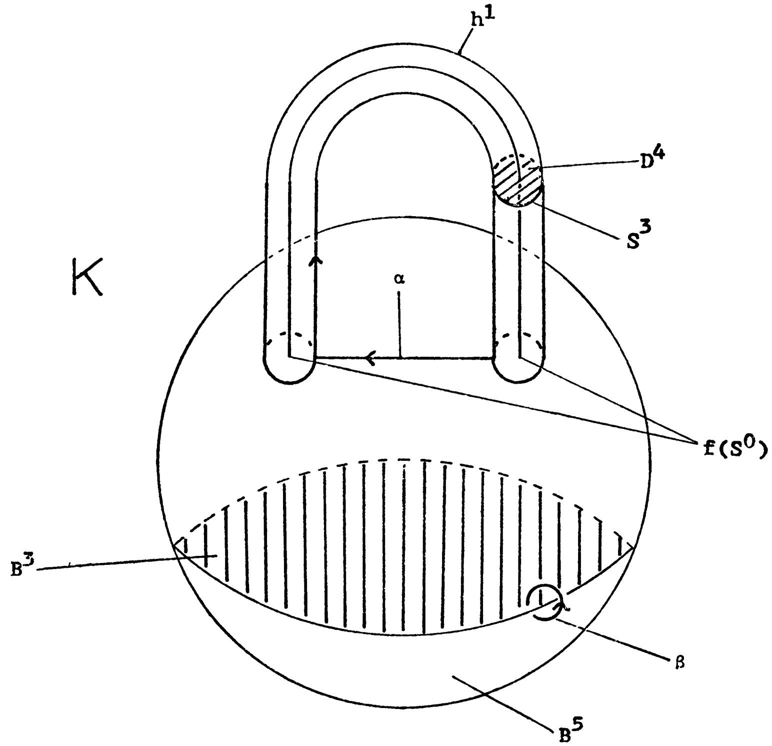


FIGURE 1

Let $\approx$ denote diffeomorphism, $\simeq$ denote homotopy equivalence, and $\vee$ denote wedge product. Then

$$K \approx S^1 \times D^4 - B^3 \quad K \simeq S^1 \vee S^3,$$
$$\partial K \approx S^1 \times S^3 - S^2 \quad \partial K \simeq S^1 \vee S^1 \vee S^3.$$

As in Figure 1, let α denote the generator of $\pi_1(\partial K)$ which goes around the handle and β the generator which links once the "flat" S^2 in ∂K. Let $g: S^1 \to \partial K$ be the smooth embedding given in Figure 2. $g(S^1)$ is an embedding in the homotopy class of $\alpha^2\beta\alpha^{-1}\beta^{-1} \in \pi_1(\partial K)$. If $*$ is

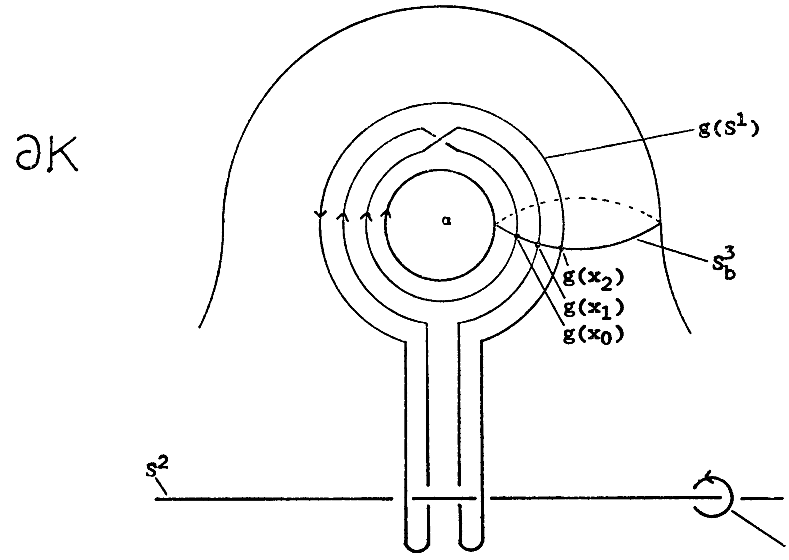


FIGURE 2

the base point of S^3, then in the product structure of $\partial K \approx S^1 \times S^3 - S^2$, $S^1 \times *$ is an embedding in the homotopy class of α, and we will henceforth identify α with $S^1 \times *$. Now from Figure 2, $g(S^1)$ is clearly diffeotopic to α in $\partial K \cup S^2$. Since α has a trivial normal bundle in ∂K then so does $g(S^1)$. Add a 2-handle to K by g, rounding corners to obtain $B = K \cup_g h^2 = B * \cup_f h^1 \cup_g h^2$. Now $B \cup B^3 = B^* \cup B^3 \cup_f h^1 \cup_g h^2 \approx B^5 \cup_f h^1 \cup_\alpha h^2 \approx B^5$. The last diffeomorphism is from the handle cancellation theorem, since the attaching sphere of the 2-handle intersects the belt sphere of the 1-handle transversely at one point. Therefore B is a knot complement; that is $B = B^5 - k'B^3$ for some smooth proper embedding $k': B^3 \to B^5$. Furthermore $\partial B = S = S^4 - kS^2$ is the complement of a smooth slice knot, where $k = k' | \partial B^3$.

III. **Homotopy calculations.** We have $B \simeq S^1 \vee S^1 \cup D^2$, and $\pi_1(B)$

$=(a, \beta \mid \alpha^2\beta\alpha^{-1}\beta^{-1})$. By considering the trace of the surgery associated with the addition of the 2-handle to K, we see that the inclusion $i: S \to B$ induces the isomorphism

$$i_*: \pi_1(S) \xrightarrow{\cong} \pi_1(B).$$

That is, let W be the trace (Figure 3), such that $\partial_- W = \partial K$ and $\partial_+ W = S$. Now if D^2 represents the core of h^2 and D^3 represents the transverse disc of h^2, then $\partial K \cup D^2 \simeq W \simeq S \cup D^3$. So $\pi_1(S) \cong \pi_1(\partial K \cup D^2)$. By considering the trace of the surgery associated with the addition of the 1-handle to B^*, we can see that $\pi_1(\partial K \cup D^2) \cong \pi_1(K \cup D^2) = \pi_1(B)$.

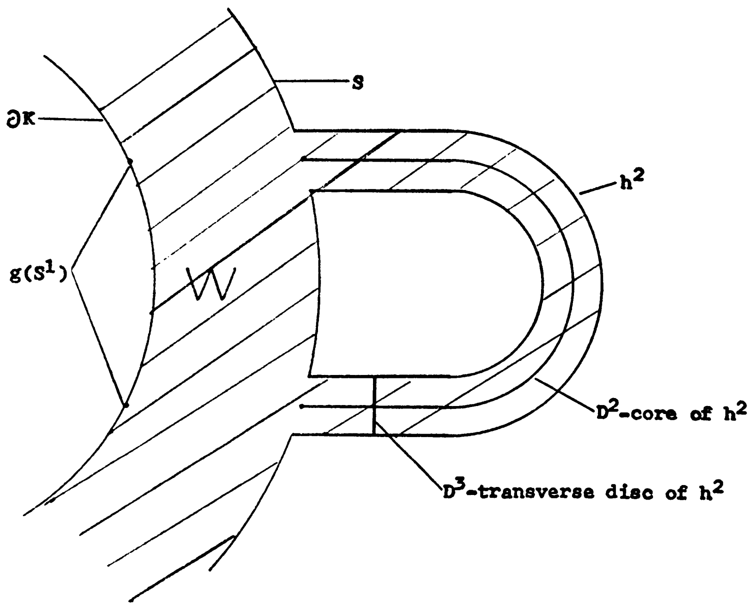


FIGURE 3

We will now consider the structure of $\pi_2(S)$ as a module over Γ. Let b be the base point of S^1. In the product structure of ∂K let $S_b^3 = b \times S^3$ be the belt sphere of h^1. [See Figure 2.] Suppose that $x_0, x_1, x_2 \in S^1$ are 3 distinct points such that $S_b^3 \cap g(S^1) = g(x_0) \cup g(x_1) \cup g(x_2)$. Let $G: S^1 \times D^3 \to \partial K$ be a trivialization of the tubular neighborhood of $g(S^1)$ in ∂K such that $G(S^1 \times D^3) \cap S_b^3 = G(x_0 \times D^3) \cup G(x_1 \times D^3) \cup G(x_2 \times D^3)$. Now $\partial(S_b^3 \cap S) = d_0 \cup d_1 \cup d_2$ where $d_i = \partial G(x_i \times D^3)$, $i = 0, 1, 2$ are embedded 2-spheres. That is, $(S_b^3 \cap S)$ is a punctured D^3. Figure 4 shows the situation in ∂K near S_b^3. The hatched area is $S_b^3 \cap S$. With choice of paths to the base point $*$ as given in Figure 4, $d_i \in \pi_2(S)$. Furthermore, $d_1 = \alpha^{-1} \cdot d_0$ where $\cdot$ denotes $\pi_1(S)$-action on $\pi_2(S)$. This can be

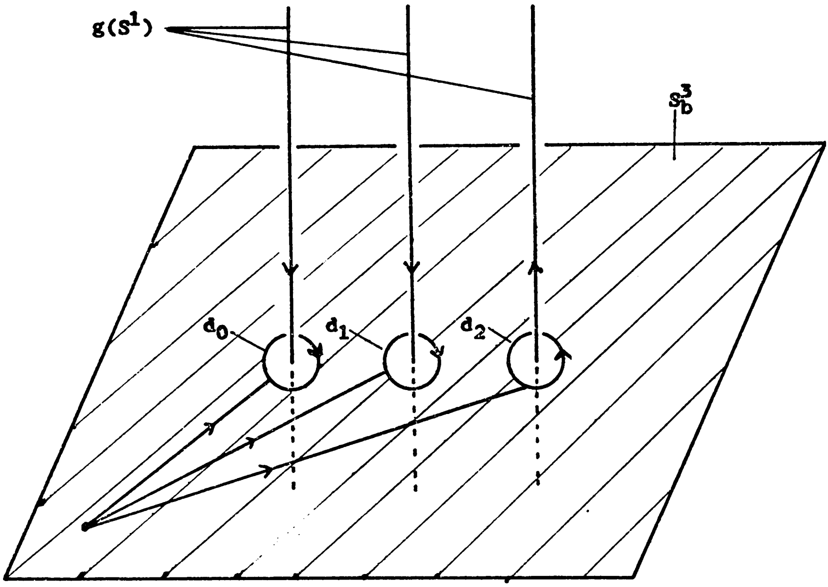


FIGURE 4

seen by moving d_0 around the boundary of the tubular neighborhood of $g(S^1)$ until it coincides (except for the path to $*$) with d_1, and then comparing the resulting paths to the base point under $\pi_1(S)$-action. Similarly, $d_2 = \beta^{-1} \cdot d_1 = \beta^{-1} \cdot (\alpha^{-1} \cdot d_0)$. Now changing the orientation of d_2, we can pipe the boundary components of $S_b^3 \cap S$ together inside $S_b^3 \cap S$ (Figure 5), obtaining an embedding in the homotopy class of $d_0 + d_1 - d_2 = (1 + \alpha^{-1} - \beta^{-1}\alpha^{-1}) \cdot d_0 \in \pi_2(S)$. Clearly this sphere bounds an embedded D^3, so $(1 + \alpha^{-1} - \beta^{-1}\alpha^{-1}) \cdot d_0 = 0 \in \pi_2(S)$. As will be shown later, $d_0 \neq 0$ in $\pi_2(S)$ because $\tilde{d}_0$, a lift of d_0 in the infinite cyclic cover $\tilde{S}$, generates $H_2(\tilde{S};\ Z) \neq 0$ as a Λ-module. Furthermore $\bar{h}_1(1 + \alpha^{-1} - \beta^{-1}\alpha^{-1}) = 2 - t^{-1} \neq 0$ in Λ, so $(1 + \alpha^{-1} - \beta^{-1}\alpha^{-1}) \neq 0$ in Γ. Hence $\pi_2(S)$ has Γ-torsion.

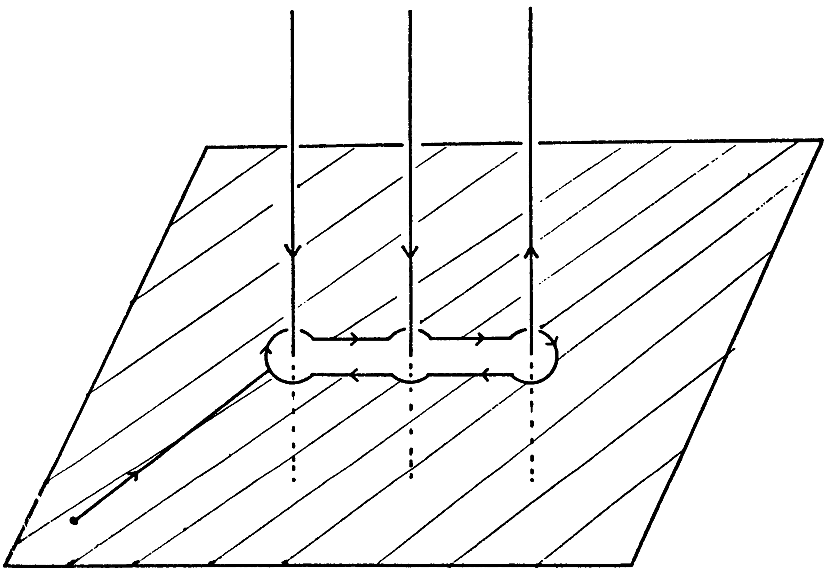
More generally, if $F(t) \in \Lambda$ and $F(t) = \sum_{i=0}^{m} a_i t^i$ $(a_0 > 0)$ then the construction proceeds exactly as before, with the exception that the 2-handle is attached by the element

$$\alpha^{a_0}\beta\alpha^{a_1}\beta\alpha^{a_2}\beta \cdots \beta\alpha^{a_m}\beta^{-m} \in \pi_1(\partial K).$$

(This is the formula produced in [7].) In this case the element $w \in \Gamma$ producing the torsion is

$$w = \sum_{k=0}^{a_0-1} \alpha^{-k} + \operatorname{sgn} a_1\left[\left(\sum_{k=0}^{a_1-1} \alpha^{-k}\right)\beta^{-1}\alpha^{a_0-1}\right] + \cdots$$
$$+ \operatorname{sgn} a_m\left[\left(\sum_{k=0}^{|a_m|-1} \alpha^{-k}\right)\beta^{-1}\alpha^{|a_{m-1}|-1}\beta^{-1} \cdots \beta^{-1}\alpha^{a_0-1}\right]$$

with the convention that if $a_q = 0$ then $\operatorname{sgn} a_q = 0$ and $\alpha^{|a_q|-1} = 1$.



$d_0 + d_1 - d_2$

FIGURE 5

IV. **The counterexample.** Giffen [2, pp. 191] has made the following statement (restated for the smooth case):

"Let $S = S^4 - k(S^2)$ be a smooth knot complement. Then $\pi_2(S)$ is the free Abelian group generated by the symbols a_c, where $1 \neq c \in [\pi_1(S), \pi_1(S)]$ the commutator subgroup of $\pi_1(S)$. The action of $\pi_1(S)$ on $\pi_2(S)$ is that induced by basis permutation

$$P_h: a_c \longrightarrow a_{hch^{-1}}$$

for $h \in \pi_1(S)$."

Consider the knot produced in detail in the special case of Theorem 1. If the above statement is correct, then $d_0 \in \pi_2(S)$ has an expansion as a finite linear combination of nontrivial elements of $[\pi_1(S), \pi_1(S)]$ (dropping a_c and keeping c), that is $d_0 = \sum_{i=1}^q m_i \xi_i$ where $\{m_i\}$ are integers and $\xi_i \in [\pi_1(S), \pi_1(S)]$. Further $\xi_i \neq \xi_j$ for $i \neq j$, and $m_i \neq 0$ for all i. It turns out in this case that $[\pi_1(S), \pi_1(S)]$ is Abelian and $\alpha^{-1} \in [\pi_1(S), \pi_1(S)]$. This will be proved in the next section. Allowing this, we compute the torsion relation in terms of the basis and the action: $\alpha^{-1} \in [\pi_1(S), \pi_1(S)]$ and action by conjugation means that the action of α^{-1} on $\pi_2(S)$ is trivial, so

$$(1 + \alpha^{-1} - \beta^{-1}\alpha^{-1}) \cdot d_0 = (2 - \beta^{-1}) \cdot d_0 = 0.$$

Or

$$2 \sum_{i=1}^{q} m_i \xi_i - \sum_{i=1}^{q} m_i \beta^{-1} \xi_i \beta = 0. \tag{1}$$

Since the action of $\pi_1(S)$ is by permutation on the basis elements, (1) can hold iff the action of β^{-1} on $\{\xi_i\}_{i=1}^q$ is permutation within the set. So we can take it instead to be a permutation ρ on the indexing set of the coefficients—that is if $\beta^{-1}\xi_i\beta = \xi_j$ then $m_i\beta^{-1}\xi_i\beta$ becomes $m_{\rho(j)}\xi_j$. So (1) is true iff the coefficients in each coordinate are zero; or $2m_i - m_{\rho(i)} = 0$ for all i. But, as a permutation on q objects, ρ has finite order, say $r \geqq 1$. Then $m_1 = (1/2)m_{\rho(1)} = (1/4)m_{\rho^2(1)} = \cdots = (1/2)^r m_{\rho^r(1)} = (1/2)^r m_1$. Since $m_1 \neq 0$ then $1 = (1/2)^r$, a contradiction.

What seems to be wrong with Giffen's statement is the description of the action of $\pi_1(S)$ on $\pi_2(S)$. For a correct description of the structure of $\pi_2(S)$ as a Γ-module in the case of fibered knots, see [1].

V. **The infinite cyclic cover** $\tilde{S}$. In order to complete the proof of Theorem 1, we will construct the infinite cyclic covering space $\tilde{S}$ of S, and study its homotopy and homology. We will produce a handlebody decomposition for $\tilde{B}$ (the infinite cyclic cover of B), and the induced covering of S will be $\tilde{S}$. The unknotted open complement $B^* = B^5 - B^3$ fibers over S^1 with fiber $\hat{D}^4$, a half-closed D^4. That is, $\hat{D}^4$ has a closed D^3 (corresponding to the unknotted B^3) removed from its boundary. So the infinite cyclic cover of B^* is $\tilde{B}^* = R^1 \times \hat{D}^4$. The attaching sphere $f(S^0)$ of h^1 lifts to a family of embedded spheres $f_i: S^0 \to \partial\tilde{B}^*$, where i is associated with $t^i \in J(t)$—the infinite cyclic group of covering transformations of $\tilde{B}^*$. Add the family of 1-handles $\{h_i^1\}$ to $\tilde{B}^*$ by f_i to get $\tilde{K}$, an infinite cyclic cover of K. Repeat the process for the 2-handles to get $\tilde{B}$. Figure 6 shows $\partial\tilde{K}$ with the family $\{g_i\}$ of attaching spheres for the $\{h_i^2\}$. Let $\tilde{S}_0=\partial\tilde{K}$, and $\tilde{S}=\partial\tilde{B}$ be the induced infinite cyclic cover of S. All integral homology groups

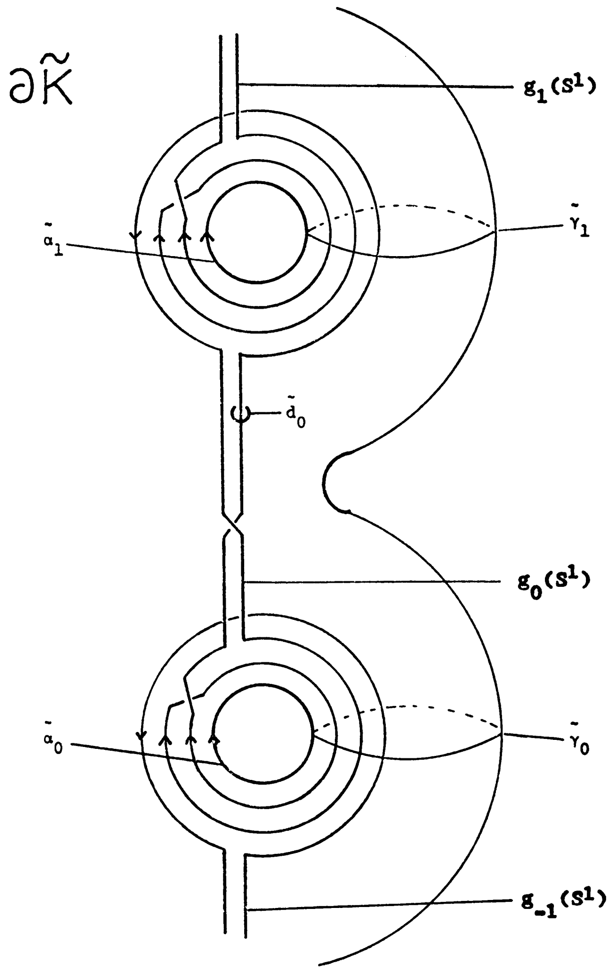


FIGURE 6

in this situation are finitely-generated Λ-modules, and the exact sequences which follow are taken to be exact sequences of Λ-modules. The calculations follow those in [5], [8].

$$H_i(\tilde{S}_0) = \Lambda \quad i = 1, 3,$$
$$= 0 \quad \text{otherwise.}$$

Let $G_i\colon S^1\times D^3\to\tilde{S}_0$ be a trivialization of the tubular neighborhood of the attaching sphere $g_i(S^1)$ of h_i^2, where G_i is a lift of $G\colon S^1\times D^3\to\partial K$. Let $\tilde{S}_1=\overline{\tilde{S}_0-\bigcup_i G_i(S^1\times D^3)}$ where upper bar denotes topological closure. By excision

$$\begin{aligned} H_i(\tilde{S}_0,\tilde{S}_1) &= \Lambda \quad i = 3, 4,\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Likewise

$$\begin{aligned} H_i(\tilde{S},\tilde{S}_1) &= \Lambda \quad i = 2, 4,\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Let $\Lambda(\tilde{\alpha})$ denote the free Λ-module generated by the symbol $\tilde{\alpha}$. Let $\tilde{\alpha}_0$ denote the lift of α at level t^0 in $\tilde{S}$ [Figure 6]. Let $\tilde{\gamma}_0$ be the lift of S_b^3 at level t^0 in $\tilde{S}$. Let $c_0=G(x_0\times D^3)$, $\tilde{c}_0=G_0(x_0\times D^3)$ and $\tilde{d}_0=\partial\tilde{c}_0=G_0(x_0\times S^2)$, the lift of d_0. With this choice of generators we have $H_1(\tilde{S}_0)=\Lambda(\tilde{\alpha}_0)$, $H_3(\tilde{S}_0)=\Lambda(\tilde{\gamma}_0)$, $H_3(\tilde{S}_0,\ \tilde{S}_1)=\Lambda(\tilde{c}_0)$, $H_2(\tilde{S},\ \tilde{S}_1)=\Lambda(\tilde{e}_0)$ where $\tilde{e}_0$ is the lift at level t^0 of the image of the core of h^2 after it has been pushed out onto the boundary of the handle.

Consider the exact sequence for the pair $(\tilde{S}_0,\ \tilde{S}_1)$:

$$\begin{tikzcd} H_3(\tilde{S}_0) \arrow[r, "\cdot\{g_i\}"] \arrow[d, equal] & H_3(\tilde{S}_0,\tilde{S}_2) \arrow[r, "\partial"] \arrow[d, equal] & H_2(\tilde{S}_1) \arrow[r] & 0 \\ \Lambda(\tilde{\gamma}_0) & \Lambda(\tilde{c}_0) & & \end{tikzcd}$$

where the homomorphism $\cdot\{g_i\}$ is as follows:

$$\tilde{\gamma}_0 \xrightarrow{\cdot\{g_i\}} \left[\sum_{t^i\in J(t)} t^i\langle\tilde{\gamma}_0\cdot g_i\rangle\right]\tilde{c}_0$$

where $\langle\tilde{\gamma}_0\cdot g_i\rangle$ is the intersection number of $\tilde{\gamma}_0$ and $g_i(S^1)$. (See [**4**, p. 515].) In this case $\cdot\{g_i\}(\tilde{\gamma}_0)=F(t^{-1})\tilde{c}_0$. So $\cdot\{g_i\}$ is a monomorphism, and $H_2(\tilde{S}_1)$ is presented as a Λ-module by the 1×1 matrix $F(t^{-1})$. Moreover, the Λ-generator of $H_2(\tilde{S}_1)$ is $\partial\tilde{c}_0=\tilde{d}_0$.

Consider now the exact sequence for the pair $(\tilde{S},\ \tilde{S}_1)$:

$$\begin{tikzcd} 0 \arrow[r] & H_2(\tilde{S}_1) \arrow[r] & H_2(\tilde{S}) \arrow[r] & H_2(\tilde{S},\tilde{S}_1) \arrow[r, "\partial"] \arrow[d, equal] & H_1(\tilde{S}_1) \arrow[d, "\cong"] \\ & & & \Lambda(\tilde{e}_0) & H_1(\tilde{S}_0) \arrow[d, equal] \\ & & & & \Lambda(\tilde{\alpha}_0) \end{tikzcd}$$

Since $\partial \tilde{e}_0 = F(t)\tilde{\alpha}_0$, then ∂ is a monomorphism and $H_2(\tilde{S}_1) \cong H_2(\tilde{S})$. Therefore $\tilde{d}_0 \neq 0$ in $H_2(\tilde{S})$; and hence $d_0 \neq 0$ in $\pi_2(S)$. This completes the proof of Theorem 1.

Since $\tilde{S}$ is the infinite cyclic cover of S, then $\pi_1(\tilde{S}) \cong [\pi_1(S), \pi_1(S)]$. From Figure 6 for $F(t) = (2-t)$, we see that $\pi_1(\tilde{S})$ is isomorphic to the group on infinitely many generators $\{\tilde{\alpha}_i\}$, with infinitely many relations $\{\tilde{\alpha}_i^2 \tilde{\alpha}_{i+1}^{-1} = 1\}$, which is Abelian because the generators commute. Furthermore, with correct choice of base point in $\tilde{S}$, $\tilde{\alpha}_0$ is the lift of $\alpha \in \pi_1(S)$, hence $\alpha^{-1} \in [\pi_1(S), \pi_1(S)]$. This completes the counterexample.

Theorem 2 is proved by construction in much the same way as Theorem 1, and is in fact easier to do when $p > 1$. If $p = 1$, the construction is exactly the same as in Theorem 1, only it takes place on the unknotted ball pair (B^{n+3}, B^{n+1}). If $p > 1$, then attach h^p to $B^* = B^{n+3} - B^{n+1}$ by a nullisotopic S^{p-1} in ∂B^*, obtaining $K = B^* \cup h^p$. As before

$$K \approx S^p \times D^{n+3-p} - B^{n+1} \qquad K \simeq S^1 \vee S^p$$

$$\partial K \approx S^p \times S^{n+2-p} - S^n \qquad \partial K \simeq S^1 \vee S^p \vee S^{n+2-p}.$$

Attach h^{p+1} to K by an embedding in the homotopy class of $F(t) \cdot \alpha$, where α generates $\pi_p(\partial K)$. The knot produced has all the characteristics listed in Theorem 2.

Theorem 3 is proved in exactly the same way as Theorem 2. The sphere pairs constructed in Theorem 2 bound ball pairs. These ball pairs have the characteristics listed in Theorem 3. However, the hypothesis of Theorem 3 allows $(n+1)/2 \leqq p \leqq n$. For p in this range (in and above the middle dimension of the boundary sphere pair), the construction can still be done. (See [8].)

Theorem 3 provides a generalization of the results of T. M. Price [6]. Using Zeeman's twist-spinning, he produces a $(B^6, k'B^4)$ with boundary (S^5, kS^3) such that $\pi_1(B^6 - k'(B^4)) = Z$ but $\pi_1(S^5 - k(S^3)) \neq Z$. Theorem 3 improves this example by one dimension, i.e. $(S^4, kS^2) = \partial(B^5, k'B^3)$ such that $\pi_1(B^5 - k'(B^3)) = Z$, $\pi_1(S^4 - k(S^2)) \neq Z$. Furthermore, $\pi_1(S^4 - k(S^2))$ and $\pi_2(B^5 - k'(B^3))$ can be specified as in Theorems 2 and 3.

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