0. Introduction. The Gelfand-Mazur theorem for commutative Banach algebras may be stated as follows: If $A$ is a commutative Banach algebra and if $M_1$ and $M_2$ are two regular maximal ideals of $A$ then $A/M_1$ and $A/M_2$ are $C$-isomorphic.

A class of not necessarily commutative Banach algebras generalizing commutative Banach algebras is motivated by the above result and is defined as follows:

A Banach algebra $A$ is called a $Q$-uniform Banach algebra if:

(i) $Q$ is a simple Banach algebra with identity $e_Q$.

(ii) $A$ is a $Q$-bimodule. Thus for $a, b \in A$, $q, q' \in Q$ $qa$ and $aq$ are (not necessarily) equal elements of $A$; $(q, a) \rightarrow qa$, $(q, a) \rightarrow aq$ are bilinear;

$$e_Qa = ae_Q = a; q(ab) = (qa)b, (ab)q = a(bq), (qa)q' = q(aq'), q(q'a) = (qq')a, a(qq') = (aq)q'; ||qa||, ||a|| \leq ||a|| ||q||.$$

(iii) If $M$ is a regular maximal ideal of $A$ then $A/M$ is $C$-isomorphic to $Q$. (Alternatively, if $\eta: A \rightarrow Q_1$ is a $C$-epimorphism of $A$ onto a simple Banach algebra $Q_1$ with identity, then $Q_1$ is $C$-isomorphic to $Q$.)

A class of $Q$-uniform algebras may be described as follows. Let $Q$ be an arbitrary simple Banach algebra with identity $e_Q$, e.g., $Q = \text{End}_C(C^n) \equiv$ the set of $C$-endomorphisms of $C^n$. Let $X$ be a compact Hausdorff space and let $A = C(X, Q) \equiv$ the set of $Q$-valued continuous functions on $X$. Clearly $A$ is a $Q$-bimodule as defined in (ii) above. We verify (iii) in the next paragraph.

Let $M_A \subseteq \mathfrak{M}_A \equiv$ the set of regular maximal ideals of $A$. We show that there is an $x_0 \in X$ such that $M_A = \{f: f(x_0) = 0\}$. Otherwise there is for each $x$ an $f_x \in M_A$ such that $q_x = f_x(x) \neq 0$. The ideal generated in $Q$ by $q_x$ must be $Q$ since $Q$ is simple and so if $\sum_{i=1}^{n} q_x q_i = e_q$ then
\( j = \sum_{i=1}^{n} q_i f_i q_i \) is such that \( j(x) = e_Q \). Via compactness of \( X \) we conclude there are open sets \( \{ U_i \}^n, \bigcup U_i = X \) and functions \( \{ f_i \} \subset M_A \) (cf. Lemma 1.1) such that for \( x \in U_i \), \( \| f_i(x) - e_Q \|_Q < \frac{1}{2} \) whence \( (f_i(x))^{-1} \) exists for \( x \in U_i \). If \( \{ \phi_i \}^n \) is a \( C \)-valued partition of unity subordinate to \( \{ U_i \}^n \), then \( \phi_i e_Q \in A, \phi_i e_Q f_i = \phi_i f_i \in M_A \) and \( \sum_{i=1}^{n} \phi_i f_i \in M_A \). Then for any \( x \)

\[
\left\| \sum_{i=1}^{n} \phi_i(x) f_i(x) - e_Q \right\|_Q = \left\| \sum_{i=1}^{n} \phi_i(x) (f_i(x) - e_Q) \right\|_Q \\
\leq \sum_{i=1}^{n} | \phi_i(x) | \| f_i(x) - e_Q \|_Q.
\]

However

\[
| \phi_i(x) | \| f_i(x) - e_Q \|_Q < \frac{1}{2} | \phi_i(x) | = \frac{1}{2} \phi_i(x), \quad \text{if} \ x \in U_i \\
= 0 \leq \frac{1}{2} \phi_i(x), \quad \text{if} \ x \notin U_i.
\]

Thus

\[
\left\| \sum_{i=1}^{n} \phi_i(x) f_i(x) - e_Q \right\|_Q \leq \frac{1}{2} \sum_{i=1}^{n} \phi_i(x) = \frac{1}{2} < 1
\]

and we conclude

\[
\left( \sum_{i=1}^{n} \phi_i f_i \right)^{-1}
\]

exists, a contradiction. Hence for some \( x_0 \), \( M_A = \{ f : f(x_0) = 0 \} \). The map \( \eta : f \to f(x_0) \) is a \( C \)-epimorphism of \( A \) onto \( Q \) and \( \ker(\eta) = M_A \), whence \( A/M_A \) is \( C \)-isomorphic to \( Q \).

1. **Fundamentals.** In this section we gather some elementary facts about \( Q \)-uniform algebras.

**Lemma 1.1.** If an algebra \( A \) is a bimodule over an algebra \( B \) then every regular ideal \( I \) of \( A \) is a \( B \)-ideal.

**Proof.** Let \( u a - a \in I \) for all \( a \in A \). Then if \( b \in B, x \in I \) we see \( u(b x) - b x \in I \), \( (u b) \cdot x \in I \) and thus \( b x \in I \). Similarly \( x b \in I \).

**Lemma 1.2.** Let \( A \) be \( Q \)-uniform and let \( M_A \subset \mathfrak{M}_A, u/M_A = e_Q = \text{identity of } Q \cong A/M_A \).

(i) If every \( C \)-monoendomorphism \( \alpha \) of \( Q \) such that \( \alpha(e_Q) = e_Q \) is a \( C \)-automorphism, then \( A = Qu \oplus M_A \).

(ii) If \( A = Qu \oplus M_A, \) if \( \eta \in \text{Epi}_C(A, Q) \) the set of \( C \)-epimorphisms of \( A \) onto \( Q \), if \( \ker(\eta) = M_A \) and if \( \alpha_\eta(q) = \eta(qu) \) then \( \alpha_\eta \in \text{Aut}_C(Q) \) the set of \( C \)-automorphisms of \( Q \).
Remark. Every noncommutative Banach algebra $B$ with identity $e$ has a nontrivial group $\text{Aut}_C(B)$. Indeed let $x \in B \setminus \{\text{center of } B\}$ and let $\varepsilon > 0$ be such that $(e + \varepsilon x)^{-1}$ exists. Then the inner automorphism

$$y \to (e + \varepsilon x)^{-1}y(e + \varepsilon x)$$

is nontrivial.

Proof. (i) Let $\eta \in \text{Epi}_C(A, Q)$ and let $\ker(\eta) = M_A$. Define $\alpha_\eta \in \text{End}_C(Q)$ by $\alpha_\eta(q) = \eta(qu)$. Then $\alpha_\eta$ is clearly linear and $C$-homogeneous. Furthermore, $qu - uqu, uq - uqu \in M_A$, whence $qu - uq \in M_A$. Thus $\alpha_\eta(q^1 q_2) = \eta(q_1 q_2 u) = \eta(q_1 q_2 u^2) = \eta(q_1 u q_2 u) = \alpha_\eta(q_1) \alpha_\eta(q_2)$. Next $\alpha_\eta(q) = 0$ implies $qu \in M_A$. Thus $\sum_{i=1}^n q_i u q_i' \in M_A$ for all $q_i, q_i'$. However, $u q_i' = q_i'u + m_i, m_i \in M_A$, and so $\sum_{i=1}^n q_i q_i' \in M_A$. Since $\{\sum_{i=1}^n q_i q_i'\}$, if $q \neq 0$, is a nontrivial ideal in $Q$, it is $Q$ and so $Qu \subseteq M_A$. Thus $e_q u = u \in M_A$, a contradiction. Hence $\alpha_\eta(q) = 0$ implies $q = 0$ and $\alpha_\eta$ is injective. Finally, $\alpha_\eta(e_q) = \eta(e_q u) = \eta(u) = e_q$. By hypothesis, then, $\alpha_\eta \in \text{Aut}_C(Q)$.

Now for $a \in A$ consider $a - a_\eta^{-1} \eta(a)u$. Then

$$\eta(a - a_\eta^{-1} \eta(a)u) = \eta(a) - \alpha_\eta(\alpha_\eta^{-1} \eta(a)) = \eta(a) - \eta(a) = 0.$$ 

Thus $a - a_\eta^{-1} \eta(a)u = m \in M_A$. We saw in the preceding paragraph that $qu \in M_A$ implies $q = 0$, i.e., $Qu \cap M_A = \{0\}$. Thus $A = Qu \oplus M_A$.

(ii) If $A = Qu \oplus M_A$ and $\ker(\eta) = M_A$ then $\eta(Qu) = \eta(A) = Q$, whence $\alpha_\eta \in \text{Aut}_C(Q)$.

Remarks. 1. The hypothesis of Lemma 1.2 is neither too restrictive nor superfluous. Clearly, if $Q$ is finite-dimensional, the hypothesis is satisfied. As G. K. Kalisch has noted, on the other hand, let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded $C$-endomorphisms of $\mathcal{H}$. If $\{\phi_n\}$ is a complete orthonormal set in $\mathcal{H}$ the elements of $\mathcal{B}(\mathcal{H})$ may be represented by countably infinite matrices. Define $\beta: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by the rule:

$$\beta:\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \to \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \cdots \\ 0 & a_{11} & 0 & a_{12} & \cdots \\ a_{21} & 0 & a_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then $\beta$ is a monoendomorphism of $\mathcal{B}(\mathcal{H})$, $\beta(I) = I$ and $\beta$ is not epic.

Next we show that if $T$ is compact then so is $\beta(T)$. Indeed, let $x_k = \{x_k^n\}$ be a sequence converging weakly in $\mathcal{H}$ to $0$: $x_k \rightharpoonup 0$. Thus we may assume $\|x_k\|^2 = \sum_{n=1}^\infty |x_k^n|^2 \leq 1, x_k^n \to 0$ as $k \to \infty$ for each $n$. 

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Thus if $T = (t_{ij})$ and $\beta(T) = (s_{ij})$ we see that

$$\beta(T)(x_k) = \left\{ \sum_j a_{1j} x_k, z_j - 1, \sum_j a_{2j} x_k, z_j, \cdots, \sum_j a_{2p-j} x_k, z_j - 1, \cdots \right\}.$$ 

If we set $x_k' = \{x_{k1}, x_{k2}, \cdots\}$, $x_k'' = \{x_{k2}, x_{k4}, \cdots\}$ then

$$\beta(T)(x_k) = Tx_k' + Tx_k''.$$

Clearly $x_k', x_k'' \not= 0$ whence $|Tx_k' |, |Tx_k'' | \to 0$, whence $|\beta(T)(x_k)| \to 0$ and so $\beta(T)$ is compact. Furthermore

$$|\beta(T)x| \leq |T|^2 (|x'|^2 + |x''|^2) = |T|^2 |x|^2$$

i.e., $|\beta(T)| \leq |T|$, and so $\beta$ is continuous. Clearly $\beta(I) = I$.

Thus we see that if $\mathfrak{K}$ denotes the (unique) maximal ideal of compact operators in $\mathfrak{B}(\mathbb{S})$, then $\beta(\mathfrak{K}) \subset \mathfrak{K}$ whence $\beta$ may be viewed as a $C$-endomorphism $B$ of the simple quotient algebra $Q = \mathfrak{B}(\mathbb{S})/\mathfrak{K}$.

We show $B$ is a continuous monoendomorphism of $Q$, that $B(e_Q) = e_Q = I/\mathfrak{K}$, and that $B(Q) \subset \mathfrak{K}Q$. Indeed, if $q_n \in Q$, $|q_n| \to 0$, let $T_n/\mathfrak{K} = q_n$, $|T_n| \to 0$ (by virtue of the open mapping theorem). Then $B(q_n) = \beta(T_n)/\mathfrak{K} \to 0$ and so $B$ is continuous. Since $\beta(I) = I$ we see $B(e_Q) = e_Q$.

If $B(q) = 0$, let $q = T/\mathfrak{K}$. Then $\beta(T) \in \mathfrak{K}$. In the notations used earlier, let $y_k = \{x_{k1}, x_{k1}, x_{k2}, x_{k2}, x_{k3}, x_{k3}, \cdots\}$. Then $y_k \not= 0$ whence $|\beta(T)(y_k)| \to 0$. However, $|\beta(T)(y_k)|^2 = 2 |Tx_k|^2$. Thus $T \in \mathfrak{K}$ and thus $q = T/\mathfrak{K} = 0$. We conclude that $B$ is monic.

Finally we show $B$ is not epic: $B(Q) \not= Q$. Indeed, if:

$$T \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then $|T\{0, 0, \cdots, 1, 0, \cdots\}|^2 \geq 2$ and we see $T$ is not compact, $T \in \mathfrak{K}$. Clearly $T \not\in \beta(\mathfrak{B}(\mathbb{S}))$. We show $T/\mathfrak{K} \not\in B(Q)$. Indeed, if $B(q) = T/\mathfrak{K}$, let $q = S/\mathfrak{K}$. Then $B(q) = \beta(S)/\mathfrak{K}$ whence $\beta(S) - T \in \mathfrak{K}$. If $S \sim (s_{ij})$ then:
However, \(|(\beta(S) - T)\{0, 0, \cdots, 0, 1, 0, \cdots\}| \geq 1\) whence \(\beta(S) - T \in \mathfrak{F}\). The contradiction implies \(B\) is not epic. In conclusion we see that \(\mathfrak{B}(\mathfrak{S})/\mathfrak{F} = Q\) is a simple Banach algebra for which there is a monoendomorphism \(B\) carrying the identity itself and yet failing to be an automorphism.

2. If \(\tilde{u}/M_A = e_0\), let \(\tilde{\alpha}_e(q) = \eta(q\tilde{u})\). Then \(u = u + m, m \in M_A\) and \(g\tilde{u} = qu + gm\). However (Lemma 1.1) \(gm \in M_A\), whence \(\eta(q\tilde{u}) = \eta(qu)\) and thus \(\tilde{\alpha}_e = \alpha_e\). In other words the definition of \(\alpha_e\) is independent of the choice of a member of \(\mathfrak{E}^{-1}(e_0)\).

Lemma 1.3. Let \(A\) be \(Q\)-uniform but without an identity. Define \(A_e = \{qe + a: q \in Q, a \in A\}\) where \(e\) is an adjoined identity defined to satisfy \(ea = ae = a, qe = eq\). Then \(A_e\) is \(Q\)-uniform and there is a one-one correspondence between \(\mathfrak{S}_{A_e} = \{I_e\}\), the set of ideals of \(A_e\) and not contained in \(A\) and \(\mathfrak{S}_A = \{I\}\), the set of regular ideals of \(A\). In particular, there is a one-one correspondence between \(\mathfrak{S}_{A_e} \setminus \{A\}\) and \(\mathfrak{S}_A\). Furthermore, there is a one-one correspondence between \(\operatorname{Epic}(A, Q) \setminus \{\eta_e: \ker(\eta_e) = A\}\) and \(\operatorname{Epic}(A, Q)\).

Proof. Clearly \(A_e\) is a \(Q\)-bimodule. Let \(I_e \in \mathfrak{S}_{A_e}\) and \(I = I_e \cap A\). Since \(I_e \subseteq A, I \neq A\) and \(I\) is a proper ideal of \(A\). From Lemma 1.1 we know \(I_e\) is a \(Q\)-ideal. Let \(q' \neq 0\) and \(q'e + a' \in I_e\) (some such \(q'\) exists since \(I_e \subseteq A\)). We note \(x(eq) = (xe)q = x(qe) = x(eq)\), whence \(x(eq - qe) = 0\). Setting \(x = e\), we see \(eq = ge\). Thus \(\sum_{i=1}^n q_{ii}q_{i}e + \sum_{i=1}^n q_{ii}a'q_{i}e \in I_e\). Since \(Q\) is simple and \(q' \neq 0\) we see \(\sum_{i=1}^n q_{ii}q'q_{i}\) = \(Q\), and hence for any \(q\) and some \(a, qe + a \in I_e\). In particular, \(-eq + u = -e + u \in I_e\), for some \(u \in A\). Direct calculation shows \(u(a - a)\) and \(au - a \in I\) for all \(a \in A\). Hence \(I\) is a regular ideal of \(A\). By Lemma 1.1, \(I\) is a \(Q\)-ideal. Conversely, suppose \(I\) is a regular ideal in \(A\); let \(u \in A\) be an identity modulo \(I\). Set \(I_e = \{x: x \in I, ux, xu \in I\}\). Since \(u(e - u) = (e - u)u = u - u^2 \in I\), whence \(-e - u \in I_e\), we have \(I_e \subseteq A\). Moreover, \(I_e \supseteq A_e\) since \(e \in I_e\) implies \(u \in I\), whence \(a = (a - ua) + ua \in I\) for all \(a \in A\), a contradiction. \(I\) is an ideal in \(A_e\). In fact, if \(x \in I_e\) and \(y = qe + a \in A_e\), then \(u(xy) = (ux)(qe + a) = (ux)q + (ux)a \in I\); and \((xy)u = xu(qe + a) = xuq + xau = [(xu) - u(xu)] + ux(qu) + u(xa) + [(xau) - u(xau)] \in I\) and \(xy \in I_e\). Similarly, \((yx)u, u(yx) \in I\), hence
\(yx \in I_e\). Thus \(I_e\) is an ideal and by Lemma 1.1, \(I_e\) is a \(Q\)-ideal. Direct calculation shows \(I_e \cap A = I\). In fact, if \(a \in I_e \cap A\), then \(ua \in I\), and hence \(a = (a - ua) + ua \in I\). Conversely, if \(a \in I\), then \(ua = -(a - ua) + a \in I\), whence \(a \in I_e \cap A\). It follows that the correspondence \(\mathcal{A}_A \ni I_e \mapsto I_e \cap A \in \mathcal{A}_A\) is epic. We show it is one-one. Suppose \(I_e \cap A = J_e \cap A = I\) for \(I_e, J_e \in \mathcal{A}_A\). As seen above, there exist \(u, v \in A\) such that \(-e + u \in I_e, -e + v \in J_e\). Then \(-v + vu \in I\) and \(-u + vu \in I\), whence \(u - v \in I\). Suppose \(ge + a \in I\). Then \(uq + ua = u(ge + a) \in I\), \(-a + ua = -(e + u)a \in I\), and consequently, also \(uq + a = uq + ua - (a + ua)I\). But then since \(ge = eq\) and \(J_e, I\) are \(Q\)-ideals, \(ge + a = (e - v)q + (u - v)q + uq + ua \in J_e\). Hence \(I_e \subseteq J_e\). Similarly, we have \(J_e \subseteq I_e\), and thus \(I_e = J_e\). It follows that the correspondence \(\mathcal{A}_A \ni I_e \mapsto I_e \cap A = I \in \mathcal{A}_A\) is one-one.

Now, \(A\) is maximal in \(A_e\) and \(A_e/A\) is \(C\)-isomorphic to \(Q\). It follows easily from the above correspondence that there is a one-one correspondence between \(\mathfrak{M}_A \setminus \{A\}\) and \(\mathfrak{M}_A\). For if \(M_e \in \mathfrak{M}_A \setminus \{A\}\), then \(M_e \neq A\), and since \(A\) is maximal in \(A_e\), \(M_e \subseteq A\). Then \(M_A = M_e \cap A\) is a maximal regular ideal in \(A\). On the other hand, if \(M_A \in \mathfrak{M}_A\) and \(u \in A\) is such that \(u/M_A = e_Q = \text{identity of } Q \cong A/M_A\); then \(M_e = \{x : x \in A_e, xu, ux \in M_A\}\) is a maximal ideal in \(A_e\), \(M_e \neq A\), and \(M_e \cap A = M_A\).

We continue by letting \(\eta \in \text{Epi}_C(A, Q)\), \(\ker(\eta) = M_A\). Define \(\eta_e \in \text{Epi}_C(A_e, Q)\) by the formula \(\eta_e(ge + a) = \alpha_e(q) + \eta(a)\). Thus \(\ker(\eta_e)\) is a maximal ideal \(M_e\). However, \(M_e \cap A = \{a : a \in A, \eta_e(a) = 0\}\) = \(\ker(\eta) = M\). Thus \(M_e = M\) and \(A/M_e\) is \(C\)-isomorphic to \(Q\), whence \(A_e\) is \(Q\)-uniform.

If \(\eta \in \text{Epi}_C(A, Q)\) let \(\tau : \eta \rightarrow \eta_e \in \text{Epi}_C(A_e, Q)\) as defined above. Conversely, given \(\eta_e \in \text{Epi}_C(A_e, Q)\) (\(\ker(\eta_e) \neq A\)) let \(\sigma : \eta_e \rightarrow \eta \in \text{Epi}_C(A, Q)\) where \(\eta\) is the restriction of \(\eta_e\) to \(A\). We show that \(\sigma \tau = \text{identity}, \sigma \tau = \text{identity}\). Indeed, \(\tau(ge + a) = \alpha_e(q) + \eta(a)\) and \((\sigma \tau) \eta(a) = \eta(a)\). Furthermore, \(\sigma \eta_e(a) = \eta(a)\), \((\sigma \tau) \eta_e(ge + a) = \alpha_e(q) + \eta(a)\). However, writing \(u = e + m_e, m_e \in M_e\) as earlier, we find \(ge = qu - qm_e, \alpha_e(q) = \eta_e(qe) = \eta_e(qu) = \eta(qu) = \alpha_e(q)\). Thus \((\sigma \tau \eta_e)(ge + a) = \alpha_e(q) + \eta(a) = \eta_e(qe + a)\).

In the course of the above we have established not only the one-one relationship between \(\text{Epi}_C(A, Q)\) and \(\text{Epi}_C(A_e, Q) \setminus \{\eta_e : \ker(\eta_e) = A\}\) but also the formula \(\alpha_e = \alpha_{\eta_e}\) when \(\tau(\eta) = \eta_e\).

In the following we topologize various sets as follows: \(\text{Epi}_C(A, Q)\): the weak topology where a typical neighborhood is

\[
N(\eta_0) = \{\eta : \|\eta(a_i) - \eta_0(a_i)\|_Q < \epsilon, a_i \in A, i = 1, 2, \ldots, n\}.
\]

\(\text{Aut}_C(Q)\): the weak topology where a typical neighborhood is

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\[ N(\alpha_0) = \{ \alpha : \| \alpha(q_i) - \alpha_0(q_i) \|_Q < \epsilon, q_i \in Q, i = 1, 2, \ldots, n \} \]

\[ \mathcal{M}_A : \text{the strongest topology such that the map} \]

\[ \rho : \text{Epi}_C(A, Q) \ni \eta \to \ker(\eta) \in \mathcal{M}_A \]

is continuous. Direct verification shows that the first two are Hausdorff topologies.

**Lemma 1.4.** Assume that for each \( M_A \in \mathcal{M}_A \) and \( u \) such that \( u/M_A = \text{identity of } A/M_A \) the direct sum decomposition \( A = Qu \oplus M \) obtains (cf. Lemma 1.2). Then there is a bijection \( \theta \) of \( \text{Epi}_C(A, Q) \) onto \( \mathcal{M}_A \times \text{Aut}_C(Q) \). Furthermore, if \( \pi \) is the projection of \( \mathcal{M}_A \times \text{Aut}_C(Q) \) on \( \mathcal{M}_A \) then \( \rho = \pi \theta \) is a continuous open map of \( \text{Epi}_C(A, Q) \) on \( \mathcal{M}_A \).

**Proof.** For \( \eta \in \text{Epi}_C(A, Q) \) let \( \theta(\eta) = (\ker(\eta), \alpha_\eta) = (M_A, \alpha_\eta) \). Here \( \alpha_\eta(q) = \eta(qu) \) where \( A = Qu \oplus M_A \). We show first that \( \alpha_\eta \) is independent of the choice of \( u \). Indeed if \( u_1/M_A = u/M_A \), then \( u_1 = u + m \), \( m \in M_A \) and \( \eta(qu_1) = \eta(qu) + \eta(qm) \). Since \( M_A \) is regular, it is a \( Q \)-ideal (Lemma 1.1) whence \( \eta(qm) = 0 \) and thus \( \eta(qu_1) = \eta(qu) \).

The map \( \theta \) is one-one since if \( \theta(\eta_1) = \theta(\eta_2) \) then \( \ker(\eta_1) = \ker(\eta_2) = M_A \) say. If \( \eta_1 \neq \eta_2 \) for some \( qu + m, m \in M_A, \eta_1(qu + m) \neq \eta_2(qu + m) \) or \( \alpha_{\eta_1}(q) \neq \alpha_{\eta_2}(q) \), contradicting \( \theta(\eta_1) = \theta(\eta_2) \).

Let \( \tilde{\theta} : \mathcal{M}_A \times \text{Aut}_C(Q) \ni (M_A, \alpha) \to \alpha \alpha_{\eta_0}^{-1} \eta_0 \in \text{Epi}_C(A, Q) \) where \( \eta_0 \) is such that \( \ker(\eta_0) = M_A \). We show \( \tilde{\theta} \) is independent of the choice of \( \eta_0 \) and that \( \tilde{\theta} \theta = \text{identity} \) (whence \( \theta \) is surjective) and that \( \tilde{\theta} = \text{identity} \) (whence \( \tilde{\theta} \) is surjective).

Indeed, if \( \ker(\eta_1) = \ker(\eta_0) = M_A \) then for some \( \alpha, \eta_1 = \alpha \eta_0 \). Thus \( \alpha_{\eta_1}^{-1} \eta_1 = \alpha_{\eta_0}^{-1} \alpha_{\eta_0} \eta_0 = \alpha_{\eta_0}^{-1} \eta_0 \). On the other hand, \( \alpha_{\eta_1}(q) = \alpha \eta_0(qu) = \alpha \alpha_{\eta_0}(q) \), whence \( \alpha_{\eta_1}^{-1} = \alpha_{\eta_0}^{-1} \alpha^{-1} \) and so \( \alpha_{\eta_1}^{-1} = \alpha_{\eta_0}^{-1} \alpha^{-1} \alpha_{\eta_0} = \alpha_{\eta_0}^{-1} \eta_0 \).

Clearly \( \ker(\alpha_{\eta_0}^{-1} \eta_0) = \ker(\eta_0) \) and \( \alpha(\alpha_{\eta_0}^{-1} \eta_0) = \alpha \alpha_{\eta_0}^{-1} \alpha_{\eta_0} = \alpha \). Thus \( \tilde{\theta} \theta = \text{identity} \). On the other hand \( \tilde{\theta}(\ker(\eta), \alpha_\eta) = \alpha_\eta \cdot \alpha_{\eta_0}^{-1} \eta = \eta \). Thus \( \tilde{\theta} = \text{identity} \).

By definition, \( \rho \) is continuous. We prove next that \( \rho \) is open. Thus let \( V \subseteq \text{Epi}_C(A, Q) \) be open. By virtue of the definitions of the topologies involved, to show \( \rho(V) \) is open it suffices to show \( \rho^{-1}(\rho(V)) \) is open. However, \( \eta' \in \rho^{-1}(\rho(V)) \) iff \( \eta' = \alpha \eta \) for some \( \alpha \in \text{Aut}_C(Q) \) and some \( \eta \in V \). Thus \( \rho^{-1}(\rho(V)) = U \{ \alpha \in \text{Aut}_C(Q) \} \}. \) Hence we shall prove that \( \alpha \) is open and thereby establish that \( \rho \) is open.

We note that \( \rho_\alpha : \text{Epi}_C(A, Q) \ni \eta \to \alpha \eta \in \text{Epi}_C(A, Q) \) is one-one since if \( \alpha \eta_1(a) = \alpha \eta_2(a) \) for all \( a \in A \), then \( \eta_1 = \eta_2 \). Next \( \rho_\alpha \) is continuous, since if

\[ N(\rho_\alpha(\eta_0)) = \{ \eta : \| \eta(a_i) - \rho_\alpha(\eta_0)(a_i) \|_Q < \epsilon, i = 1, 2, \ldots, n \} \]
Let

\[ N(\eta_0) = \{ \eta : \| \eta(a_i) - \eta_0(a_i) \|_q < \epsilon / \| \alpha \|, \ i = 1, 2, \cdots, n \}. \]

Then for \( \eta \in N(\eta_0) \) we find \( \rho_\alpha(\eta) \in N(\rho_\alpha(\eta_0)). \) Hence \( \rho_\alpha \) is continuous. Since \( \rho_\alpha(\alpha^{-1}\eta) = \eta \) we see \( \rho_\alpha \) is surjective and clearly \( \rho_\alpha^{-1} = \rho_{\alpha^{-1}} \), whence \( \rho_\alpha^{-1} \) is continuous. We conclude \( \rho_\alpha \) is a self-homeomorphism (automorphism) of \( \text{Epi}_C(A, Q) \). Since \( \rho_\alpha(V) = \alpha V \) we see \( \alpha V \) is open and thus the map \( \rho \) is open.

**Lemma 1.5.** If \( A \) has an identity \( e_A \) the map \( \tau : \eta \to \alpha_\eta \) is continuous in the topologies considered.

**Proof.** Let \( \alpha_{\eta_0} = \tau(\eta_0) \) be given and let

\[ N(\alpha_{\eta_0}) = \{ \alpha : \| \alpha(q_i) - \alpha_{\eta_0}(q_i) \| < \epsilon, \ i = 1, 2, \cdots, n \} \]

be given. Let \( U(\eta_0) = \{ \eta : \| \eta(q_i e_A) - \eta_0(q_i e_A) \| < \epsilon, \ i = 1, 2, \cdots, n \} \).

Then for \( \eta \in U(\eta_0) \)

\[ \| \alpha(q_i) - \alpha_{\eta_0}(q_i) \| = \| \eta(q_i e_A) - \eta_0(q_i e_A) \| < \epsilon, \]

whence \( \tau(U(\eta_0)) \subseteq N(\alpha_{\eta_0}) \), and the continuity of \( \tau \) is established.

**Corollary.** If \( A \) has an identity \( e_A \) the map \( \theta \) is continuous.

**Proof.** Indeed, \( \theta(\eta) = (\rho(\eta), \tau(\eta)) \) whence, since \( \rho \) and \( \tau \) are continuous, \( \theta \) is continuous.

Any general statement asserting the compactness of \( \text{Epi}_C(A, Q) \) is false. This follows from the fact, for some \( Q \) and \( A \), e.g., \( Q = \text{End}_C(C^n) \equiv \) the set of linear endomorphisms of \( C^n \) and \( A = C(X, Q) \) where \( X \) is compact Hausdorff, that the function \( \delta(\eta) \) may be unbounded, for some \( a \), where \( \delta(\eta) \) is the function defined on \( \text{Epi}_C(A, Q) \) by the formula \( \delta(\eta) = \eta(a) \).

Indeed, \( \text{Aut}_C(\text{End}_C(C^n)) \) is the set of inner automorphisms \( \alpha_S \) given by

\[ \alpha_S : q \to S^{-1} q S \]

and \( S \in \text{Gl}(n, C) \). If \( n = 2 \) the following calculations show that one may choose \( S \) so that \( \| \alpha_S \| \) (operator norm) is arbitrarily large (or small).

For \( N > 0 \), let numbers \( u, v, w, x \) be chosen so that \( uvwx \neq 0 \) and let

\[ q = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, \]
Then if $\lambda \mu \neq 0$ and
\[ S = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \]
we find
\[ \alpha_s(q) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \mu \\ x \end{pmatrix} + \begin{pmatrix} \nu \mu \lambda^{-1} \\ \nu \mu^{-1} \lambda \end{pmatrix}. \]

Set $\lambda = 1$ and choose $\mu$ so that
\[ \|\alpha_s(q) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \| > N \|q\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \| . \]

In this case $\|\alpha_s\| > N$. Similarly, if $q' = \alpha_s q$, then
\[ \|\alpha_s^{-1} q'\|, \quad \frac{\|\alpha_s^{-1} q'\|}{\|q'\|} = \frac{\|q\|}{\|q'\|} < \frac{\|q\|}{N\|q\|} = \frac{1}{N}. \]

Hence $\|\alpha_s^{-1}\| < 1/N$. Thus $0 = \inf \{\|\alpha_s\|\} < \sup \{\|\alpha_s\|\} = +\infty$. The set \( \{q: \sup_{\alpha_s}\|\alpha_s q\| < +\infty\} \) must be of the first category and thus the set \( \{q: \sup_{\alpha_s}\|\alpha_s q\| = +\infty\} \) is of the second category and nonempty. For a $q$ such that $\sup_{\alpha_s}\|\alpha_s q\| = +\infty$, choose an $a_0 \in A$ and an $\eta_0$ such that $\eta_0(a_0) = q$. Then $\sup_{\eta_0}\|\alpha(a)\| \geq \sup_{\alpha_s}\|\alpha_s q\| = +\infty$.

2. **Special cases.** If $Q$ is a simple commutative Banach algebra with identity, then $Q$ is isomorphic to $C$. A parallel of this elementary fact is the

**Proposition 1.** If $A$ is a $Q$-uniform semisimple Banach algebra and if $A$ is not commutative then $Q$ is not commutative.

**Proof.** Let $a_1, a_2 \in A$, $a_1a_2 - a_2a_1 \neq 0$. Then since $A$ is semisimple, there is some regular maximal ideal $M$ such that $a_1a_2 - a_2a_1 \in M$, i.e.

\[ a_1/M \cdot a_2/M = a_2/M \cdot a_1/M \neq 0. \]

If $Q$ is commutative, the last relation cannot obtain.

**Corollary.** If $G$ is a locally compact group and if $L^1(G)$ is semisimple (e.g., if $G$ is abelian or compact) and $Q$-uniform, then $G$ is abelian (whence $Q \cong C$).
Proof. Define the map $h: L^1(G) \rightarrow \mathbb{C}$ by

$$h(f) = \int_G f dx.$$ 

Then $h \in \text{Epi}_C(L^1(G), \mathbb{C})$ and thus $L^1(G)/\ker(h) \cong \mathbb{C}$. Thus $\mathbb{Q} \cong \mathbb{C}$, since we have assumed $L^1(G)$ is $Q$-uniform. Thus $L^1(G)$ must be commutative by Proposition 1 and therefore $G$ is abelian.

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