A CLASS OF COUNTABLY PARACOMPACT SPACES

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A space X is said to have property $\mathfrak{B}$ if for any well-ordered monotone decreasing family $\{H_a \mid a \in A\}$ of closed sets with no common part, there is a monotone decreasing family of domains $\{D_a \mid a \in A\}$ such that

(i) $H_a \subseteq D_a$ for each $a$ in $A$ and
(ii) $\{\text{cl}(D_a) \mid a \in A\}$ has no common part.

It is shown that property $\mathfrak{B}$ characterizes the separable $T_3$-spaces that are Lindelöf and the countably compact spaces that are compact. Also, it is shown that the $T_3$-space $X$ is Lindelöf if and only if $X$ has property $\mathfrak{B}$ and every uncountable subset of $X$ has a limit point.

Throughout this paper, topological spaces are assumed to be $T_1$-spaces.

1. Preliminary results and lemmas.

1.1. If $X$ has property $\mathfrak{B}$, then $X$ is countably paracompact.

This is immediate from [1] where it is shown that $X$ is countably paracompact if and only if for any countable decreasing sequence of closed sets $\{H_n\}$ with no common part, there is a monotone decreasing sequence of domains $\{D_n\}$ such that

(i) for each $n$, $H_n \subseteq D_n$ and
(ii) $\{\text{cl}(D_n)\}$ has no common part.

1.2. If $X$ is paracompact, then $X$ has property $\mathfrak{B}$.

Proof. Let $\{a \subseteq A\}$ denote a well-ordered, monotone family of closed sets with no common part. Then $\{G_a = X - H_a \mid a \in A\}$ is an open cover of $X$. Hence, there is a locally finite open refinement $\{G'_a \mid a \in A\}$ of $\{G_a \mid a \in A\}$ such that $G'_a \subseteq G_a$ for each $a$ in $A$. For each $a$ in $A$, let $D_a = \bigcup \{G'_b \mid b \in A, b \geq a\}$. $\{D_a \mid a \in A\}$ satisfies the conditions for property $\mathfrak{B}$.

1.3. Theorem. If the $T_2$-space $X$ has property $\mathfrak{B}$, then $X$ is $T_3$.

Proof. Suppose the contrary; that is, suppose that there is a closed set $H$ and a point $P$ not in $H$ such that if $O$ is an open set containing $H$, then $P$ is in cl($O$). Let $G$ be an open cover of $H$ of minimal cardinal $\rho$ such that if $g$ is in $G$ then cl($g$) does not contain $P$. Note that it follows from the supposition that $\rho$ cannot be finite.

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Let \( \{ g_a | a \in A \} \) be a well-ordering of \( G \) according to the initial ordinal of cardinal \( \rho \). Then \( \{ h_a = H - \bigcup_{b < a} g_b | a \in A \} \) is a well-ordered monotone decreasing family of closed sets with no common part. Since \( S \) has property \( \mathfrak{B} \), there is a domain \( D \) containing \( h_{a'} \), for some \( a' \) in \( A \), such that \( P \) does not belong to \( \text{cl}(D) \). Hence, \( G' = \{ g_b | b \leq a' \} \cup D \) is an open cover of \( H \) such that if \( g \in G' \), then \( P \) is not in \( \text{cl}(g) \). But the cardinality of \( G' \) is less than \( \rho \) which is a contradiction from which the theorem follows.

The following example, brought to the attention of the author by John Mack, shows that property \( \mathfrak{B} \) cannot be replaced by countable paracompactness in Theorem 1.3.

Let \([0, \Omega)\) denote the segment of countable ordinals, where \( \Omega \) denotes the first uncountable ordinal and let \([0, \Omega] = [0, \Omega) \cup \{ \Omega \}\). Let \( Y = [0, \Omega] \times [0, \Omega] - (\Omega, \Omega) \). Then \( Y/(\Omega \times [0, \Omega]) \) is a countably compact (and therefore countably paracompact) \( T_3 \)-space that is not \( T_3 \).

Related to Theorem 1.3 are the following questions:

1. If \( X \) is a \( T_3 \)-space with property \( \mathfrak{B} \) such that each closed set is a \( G_\delta \)-set, then is \( X \) normal?
2. If \( X \) is a \( T_3 \)-space with property \( \mathfrak{B} \) such that each closed subset of \( X \) is a \( G_\delta \)-set, then is \( X \) hereditarily countably paracompact?

With techniques similar to those used in [2], Questions 1 and 2 can be shown to be equivalent.

Recall that the function \( f \) from \( X \) to \( Y \) is said to be a proper mapping if \( f \) is continuous, closed, and \( f^{-1}(y) \) is compact for each \( y \) in \( Y \).

1.4. Theorem. If \( X \) has property \( \mathfrak{B} \) and \( f \) is a proper mapping from \( X \) onto \( Y \), then \( Y \) has property \( \mathfrak{B} \).

Proof. Let \( \{ H_a | a \in A \} \) be a well-ordered, monotone decreasing family of closed sets in \( Y \) with no common part. Then \( \{ f^{-1}(H_a) | a \in A \} \) is a well-ordered monotone decreasing family of closed sets in \( Y \) with no common part. Let \( \{ D_a | a \in A \} \) denote the family of domains in \( X \) given by property \( \mathfrak{B} \) for \( \{ f^{-1}(H_a) | a \in A \} \). Let \( O_a = Y - f(X - D_a) \) for each \( a \) in \( A \). Then \( \{ O_a | a \in A \} \) is a well-ordered family of domains in \( Y \) such that \( H_a \subseteq O_a \) for each \( a \) in \( A \). Suppose that \( y \in \bigcap_{a \in A} \text{cl}(O_a) \). Then \( f^{-1}(y) \) is a compact set that intersects \( \text{cl}(D_a) \) for each \( a \) in \( A \) which is impossible; and so, \( \{ \text{cl}(O_a) | a \in A \} \) has no common part.

1.5. Corollary. If \( X \) has property \( \mathfrak{B} \) and \( Y \) is compact, then \( X \times Y \) has property \( \mathfrak{B} \).

1.6. Lemma. If \( X \) has property \( \mathfrak{B} \) and \( \{ K_a | a \in A \} \) is a well-ordered, countably centered, monotone decreasing family of closed sets in \( X \) with no common part, then there is an uncountable pair-wise disjoint family of nonempty domains and an uncountable, closed discrete subset of \( X \).
PROOF. For each $a$ in $A$, let
\[
F_a = \bigcap_{b < a} K_b \quad \text{if } a \text{ is a limit ordinal,}
\]
\[
= K_a \quad \text{otherwise.}
\]
Then \( \{ F_a \mid a \in A \} \) is a well-ordered, countably centered, monotone decreasing family of closed sets with no common part. Let \( \{ D_a \mid a \in A \} \) be the collection of domains given by property \( \mathfrak{D} \) for \( \{ F_a \mid a \in A \} \). For each $a$ in $A$, the set \( \{ b \in A \mid D_a - \text{cl}(D_b) \neq \emptyset \} \) is not empty. Let $\tau$ be the function from $A$ into $A$ that takes $a$ into the first element of \( \{ b \in A \mid D_a - \text{cl}(D_b) \neq \emptyset \} \). Observe that for each $a$ in $A$, $\tau(a) > a$.

Let $\theta$ denote the function taking $A$ into the power set of $X$ by letting $\theta(0) = D_0 - \text{cl}(D_{\tau(0)})$ and
\[
\theta(b) = D_b - \text{cl}(D_{\tau(b)}) \quad \text{if } \sup \{ \tau(a) \mid a < b \} \leq b,
\]
\[
= \emptyset \quad \text{otherwise.}
\]
Clearly $\theta(A)$ is a pair-wise disjoint collection of domains. If it can be shown that $A' = \{ a \in A \mid \theta(a) \neq \emptyset \}$ is cofinal in $A$, it will follow that $\theta(A)$ is uncountable; otherwise, \( \{ F_a \mid a \in A' \} \) would be a countable subcollection of \( \{ F_a \mid a \in A \} \) with no common part. To see that $A'$ is cofinal in $A$, suppose the contrary; that is, suppose that $b = \sup A'$ is in $A$. Let $b_1 = \tau(b)$ and, proceeding by induction, let $b_{n+1} = \tau(b_n)$. The set \( \{ b_n \mid n = 1, 2, \cdots \} \) is not cofinal in $A$, since \( \{ F_b \} \) is countably centered, so $b_0 = \sup \{ b_n \} \in A$. Since $\tau$ is monotone nondecreasing, it follows that $\sup \{ \tau(a) \mid a < b_0 \} \leq b_0$; and so $b_0$ is in $A'$.

For each $a$ in $A'$, let $P_a$ be a point of $\theta(a)$. To see that \( \{ P_a \mid a \in A' \} \) has no limit point, suppose the contrary; that is, suppose that $P$ is a limit point of \( \{ P_a \mid a \in A' \} \). Since if $a$ is a limit ordinal of $A$, $F_a = \bigcap_{b < a} F_b$, there is a last element $a'$ of $A$ such that $F_{a'}$ contains $P$. Let $\mathcal{P}_1 = \{ P_a \mid a \in A', a \leq a' \}$ and $\mathcal{P}_2 = \{ P_a \mid a \in A', a > a' \}$.

Since $P$ does not belong to $F_{a' + 1}$, $P$ is not a limit point of $\mathcal{P}_2$; and so, $P$ must be a limit point of $\mathcal{P}_1$. If \( \{ a \in A' \mid a \leq a' \} \) has no last term, $D_{a'}$ is a domain containing $P$ but no points of $\mathcal{P}_1$; hence, \( \{ a \in A' \mid a \leq a' \} \) must have a last term, say $b'$. Then $D_{b'}$ is a domain containing $P$ that contains only one point of $\mathcal{P}_1$, namely $P_{b'}$, which is a contradiction from which it follows that \( \{ P_a \mid a \in A' \} \) has no limit point.

1.7. Lemma. Suppose that $X$ is a space with an uncountable, closed, discrete subspace $H$. If $X$ has property \( \mathfrak{D} \), then there are uncountably many mutually exclusive nonempty domains in $X$.

PROOF. Let $K$ denote a subcollection of $H$ with cardinality $\aleph_1$. Let \( \{ P_a \mid a \in A \} \) be a well-ordering of $K$ according to the least ordinal
of cardinal $\mathfrak{N}_1$. For each $a$ in $A$, let $K_a = \{ P_b | b \in A, b \preceq a \}$. Then 
\{ K_a | a \in A \} is a well-ordered, monotone decreasing family of closed sets with no common part. Since the cardinality of $K$ is $\mathfrak{N}_1$, 
\{ K_a | a \in A \} is countably centered; therefore, by Lemma 1.6, there is an uncountable family of mutually exclusive, nonempty domains in $X$.

A space $X$ is said to have the *Souslin property* if there is no uncountable collection of mutually exclusive nonempty domains in $X$.

1.8. **Corollary.** If the space $S$ has the Souslin property and property $\mathfrak{B}$, then every uncountable subset of $X$ has a limit point.

1.9. **Corollary.** If $X$ is a separable space with property $\mathfrak{B}$, then every uncountable subset of $X$ has a limit point.

The following result, due to W. B. Sconyers [3, Theorem 3], is stated as a lemma.

1.10. **Lemma.** The $T_3$-space $X$ is Lindelöf if and only if for each well-ordered, monotone increasing family $\mathcal{D}$ of domains covering the space, there is a countable collection of closed sets that refines $\mathcal{D}$ and covers $X$.

2. **Main results.**

2.1. **Theorem.** The $T_3$-space $X$ is Lindelöf if and only if

(i) $X$ has property $\mathfrak{B}$ and

(ii) every uncountable subset of $X$ has a limit point.

**Proof.** It is well known that if $X$ is Lindelöf, then $X$ is paracompact; and so, by 1.2, $X$ has property $\mathfrak{B}$.

Suppose that (i) and (ii) are satisfied. By Lemma 1.10, it is sufficient to show that there is a countable collection, $\{ F_n \}$, of closed sets refining $\{ D_a | a \in A \}$ and covering $X$, where $\{ D_a | a \in A \}$ is a well-ordered, monotone increasing open cover of $X$. It follows easily from Lemma 1.6 that there is a countable subset $B$ of $A$ such that $\{ D_b | b \in B \}$ covers $X$. Hence, $\{ E_b = X - D_b | b \in B \}$ is a countable, well-ordered family of closed sets with no common part. Let $\{ G_b | b \in B \}$ be the domains given for $\{ E_b | b \in B \}$ by property $\mathfrak{B}$. Then $\{ F_b = X - G_b | b \in B \}$ is the desired collection of closed sets.

2.2. **Corollary.** The countably compact $T_3$-space $X$ is compact if and only if $X$ has property $\mathfrak{B}$.

The following theorems follow immediately from Theorem 2.1, Lemma 1.7 and Corollary 1.8:

2.3. **Theorem.** If the $T_3$-space $X$ has the Souslin Property, then $X$ is Lindelöf if and only if $X$ has property $\mathfrak{B}$.
2.4. Theorem. The separable $T_\gamma$-space $X$ is Lindelöf if and only if $X$ has property 08.

References


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