ANNULUS CONJECTURE AND STABILITY OF HOMEO-
MORPHISMS IN INFINITE-DIMENSIONAL NORMED
LINEAR SPACES

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Abstract. If $E$ is an arbitrary infinite-dimensional normed linear
space, it is shown that if all homeomorphisms of $E$ onto itself are
stable, then the annulus conjecture is true for $E$. As a result, this con-
firms that the annulus conjecture for Hilbert space is true. A partial
converse is that for those spaces $E$ which have some hyperplane
homeomorphic to $E$, if the annulus conjecture is true for $E$ and if all
homeomorphisms of $E$ onto itself are isotopic to the identity, then all
homeomorphisms of $E$ onto itself are stable.

Let $E$ denote a normed linear space with zero element $\theta$, and let
$B_r(x)$ be the ball in $E$ of radius $r$ centered at $x$, $S_r(x) = \text{Bd } B_r(x)$, $B_r = B_r(\theta)$, and $S_r = S_r(\theta)$. $H(E)$ and $SH(E)$ will denote the homeomor-
phisms on $E$ and the stable homeomorphisms on $E$, respectively.

From [5] and [6] there is an inversion homeomorphism $i \in H(E)$
for $E$ infinite-dimensional such that $i(B_1) = E - \text{Int } B_1$, $i|S_1 = \text{identity}$,
and $i^{-1} = i$. It can be seen in [7] and [8] that this homeomorphism
is a useful tool in proving topological properties of infinite-dimen-
sional normed linear spaces. In this paper $i$ will be used in establishing
a type of engulfing theorem for infinite-dimensional $E$ which is some-
what analogous to Stallings' Engulfing Theorem in [9]. Brown and
Gluck in [3] showed that $SH(E) = H^+(E)$ implies the annulus con-
jecture for finite-dimensional $E$, where $H^+(E)$ is the group of orienta-
tion preserving homeomorphisms of $E$. As an application of the en-
gulfing theorem in this paper, it will be shown that $SH(E) = H(E)$
implies the annulus conjecture for infinite-dimensional $E$ (a slightly
different technique than used in [3]). As a result, since Wong in [10]
showed that $SH(l_2) = H(l_2)$, the annulus conjecture is true in Hilbert
space and is consequently true in all infinite-dimensional separable
Banach spaces, or more generally, in all infinite-dimensional separ-
able Fréchet spaces (see [4] and [1]).

By a cell in $E$ is meant a subset $C$ of $E$ such that there exists a
homeomorphism from the pair $(B_1, S_1)$ onto the pair $(C, \text{Bd } C)$. A
closed subset $K$ of $E - \text{Int } C$ is a collar of $C$ if there exists a homeomorphism $h$ from the pair $(B_2, B_1)$ onto the pair $(K \cup C, C)$ such that $h(S_2) = \text{Bd } (K \cup C)$.

$C$ is tame if there is an $h \in H(E)$ such that $h(B_1) = C$. In [7] and [8] it is shown that $C$ is tame if and only if it has a collar. By a sphere in $E$ is meant a closed homeomorph of $S_1$ in $E$. A sphere is tame if it is the boundary of a tame cell. Sanderson showed in [8] that a sphere is tame if and only if it is bicollared. $A \subset E$ is an annulus if there exists a homeomorphism $h$ from $B_2 - \text{Int } B_1$ onto $A$ such that $\text{Bd } A = h(S_1 \cup S_2)$.

The annulus conjecture for $E$ is stated in [7] as follows.

$A_B$: If $C$ is a tame cell contained in $\text{Int } B_1$, then there exists a homeomorphism $h$ of $B_1$ onto itself such that $h(B_{1/2}) = C$ and $h|S_1 = \text{identity}$.

For infinite-dimensional $E$, it can be seen by using $i$ that $A_B$ is equivalent to the statement: the region between two disjoint tame spheres is an annulus.

$h \in H(E)$ is stable if it can be written as a finite composition of elements of $H(E)$ each of which is the identity on some open subset of $E$. The following is the stability conjecture for $E$.

$S_B$: $SH(E) = H(E)$ (for $E$ finite-dimensional $SH(E) = H^+(E)$).

**Lemma 1.** If $E$ is infinite-dimensional and $C$ is a collared cell with collar $K$, then there exists a collar $K' \subset K$ of $C$ and $g \in H(E)$ such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$.

**Proof.** Let $h: (C \cup K, \text{Bd } [C \cup K]) \to (B_3, S_3)$ be a homeomorphism such that $h(C) = B_1$. Set $K' = h^{-1}(B_2 - \text{Int } B_1)$. By Lemma 6.1 of [7], there exists $f \in H(E)$ such that $f|K' \cup C = h|K' \cup C$. Let $k: B_2 - \text{Int } B_1 - \text{Int } B_{1/2}$ be a homeomorphism such that $k(S_2) = S_{1/2}$ and $k|S_1 = \text{identity}$. Define the homeomorphism $k': E - \text{Int } B_1 \to B_1 - \theta$ by letting $k'| (B_2 - \text{Int } B_1) = k$ and, for $E - B_2$, letting $f'$ map $[x: \infty)$, $x \in B_2$, linearly onto $[k(x) : \theta)$. Let $j: (B_1 - \theta \to B_1$ be a homeomorphism such that $j|(B_1 - \text{Int } B_{1/2}) = \text{identity}$. Define $g' \in H(E)$ by $g'(x) = i(x)$ if $x \in B_1$, and $g'(x) = jf'(x)$ if $x \notin E - B_1$. Finally define $g \in H(E)$ by $g = g'f$.

**Theorem 1** (Engulfing theorem). Let $E$ be infinite-dimensional, let $A \subset E$, let $C$ be a collared cell in $E$ with collar $K$ such that $(C \cup K) \cap A = \emptyset$, and let $U \subset E$ be open such that $[E - (C \cup K)] \cap U \neq \emptyset$. Then there exists $h \in H(E)$ such that $A \subset h(U)$ and $h|C = \text{identity}$.

**Proof.** By Lemma 1, let $K' \subset K$ be a collar of $C$ and $g \in H(E)$ be such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$. Choose $B_+(x) \subset g(U) \cap \text{Int } B_{1/2}$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_+(x)]$ and
Using a proof very similar to the proof of the above theorem, the following result concerning the fattening of a collar can be established.

**Theorem 2.** Let $E$ be infinite-dimensional, let $A \subset E$, and let $C$ be a collared cell in $E$ with collar $K$ such that $C \cup \text{Cl} A \neq E$. Then there exists $h \in H(E)$ such that $A \subset h(C \cup K)$ and $h|C = \text{identity}.$

**Proof.** By Lemma 1, let $K' \subset K$ be a collar of $C$ and $g \in H(E)$ be such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$. Choose $B_{ij}(x) \subset \text{Int } B_1 - g(A)$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_{ij}(x)]$ and $f| (E - B_1) = \text{identity}$. Define $h \in H(E)$ by $h = g^{-1}f^{-1}g$.

**Lemma 2.** Let $A$ be a nondense subset of $E$, and let $f \in H(E)$ and $U \subset A$ be open such that $f| U = \text{identity}$. Then there exist $f_1, f_2, f_3 \in H(E)$ and $U_1, U_2, U_3$ open subsets of $E$ such that $U_1 \subset E - A$, $U_2 \subset E - f_1(A)$, $U_3 \subset E - f_2f_1(A)$, $f = f_3f_2f_1$, and $f_i| U_i = \text{identity}$ for $i = 1, 2, 3$.

**Proof.** Let $B_{ij}(x) \subset U$ and $B_{ij}(y) \subset E - A$. Let $N > M > 0$ be such that $\text{Int } B_{ij}(y) \cap \text{Int } B_M(x) \neq \emptyset$ and $\text{Int } B_{ij}(y) \cap [E - B_N(x)] \neq \emptyset$. Define $f_1 \in H(E)$ such that $f_1[B_M(x)] = B_{ij}(x)$ and $f_1| [E - B_N(x)] = \text{identity}$. Define $f_2, f_3 \in H(E)$ by $f_2 = f$ and $f_3(x) = f_f^{-1}f^{-1}(x)$ if $x \in f[B_N(x)]$, and $f_3(x) = x$ if $x \in E - f[B_N(x)]$. Choose $U_1$ open in $B_{ij}(y) - B_N(x)$, $U_2$ open in $B_{ij}(x) - f_f[B_N(x)]$, and $U_3$ open in $f[B_{ij}(y) - B_N(x)]$.

**Lemma 3.** If $f \in H(E)$, then there exist, for $i = 1, \ldots, n$, $f_i \in H(E)$ and open $U_i \subset E - f_{i-1} \cdots f_1 f_0(B_1)$ ($f_0 = \text{identity}$) such that $f_i| U_i = \text{identity and } f = f_n \cdots f_1$.

**Proof.** Let $f = g_m \cdots g_1$ and $V_i, i = 1, \ldots, m$, be open such that $g_i| V_i = \text{identity}$. Let $k$ be such that $1 \leq k \leq m$. If $V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)] \neq \emptyset$ ($g_0 = \text{identity}$), then choose $U_{k1}$ open in $V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)]$ and let $f_{k1} = g_k$. If $V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)] = \emptyset$, then by Lemma 2, there exist $f_{k1}, f_{k2}, f_{k3} \in H(E)$, open $U_{k1} \subset E - g_{k-1} \cdots g_1g_0(B_1)$, open $U_{k2} \subset E - f_{k2}f_{k1}$ and open $U_{k3} \subset E - f_{k2}f_{k1}g_{k-1} \cdots g_1g_0(B_1)$ such that $g_k = f_{k3}f_{k2}f_{k1}$ and $f_{k1}| U_{k1} = \text{identity}, i = 1, 2, 3$. The $f_{ij}$ and $U_{ij}$ can be relabeled to give the desired $f_i$ and $U_i, i = 1, \ldots, n$.

**Addendum to Lemma 3.** In Lemma 3 the $f_i$ can further be chosen so that $f_i \cdots f_1(B_1) \subset f_{i+1} \cdots f_1(B_1)$ for $i = 1, \ldots, n - 1$.

**Proof.** Let $U'_n = U_n$ and $f'_n = f_n$. For $k$ such that $1 \leq k \leq n - 1$, define $U'_{n-k}, f'_{n-k}, U''_{n-k+1}$, and $f''_{n-k+1}$ by an inductive step as follows.
Choose \( B_\gamma(x) \subset f_{i-1} \cdots f_{n-k} f_{n-k+1} \cdots f_1(B_1) \), \( B_\delta(y) \subset f_{i-1} \cdots f_{n-k} (U_{n-k+1}) \) such that \( B_\gamma(x) \cap (B_\delta(y) \cup B_\epsilon(z)) = \emptyset \). Let \( g \in H(E) \) be such that \( g(B_1) = B_\gamma(x) \) and \( g| (B_\delta(y) \cup B_\epsilon(z)) = \text{identity} \). Define \( U_{n-k}' = f_{n-k} \cdots f_1(\text{Int } B_\delta(y)) \), \( f_{n-k}' = f_{n-k} \cdots f_1(gf_{n-k}) \), \( U_{n-k+1}' = f_{n-k} \cdots f_1(\text{Int } B_\epsilon(z)) \), and \( f_{n-k+1}' = f_{n-k+1} f_{n-k} f_{n-k}' \). Finally let \( U_{i}' = U_i' \) and \( f_i' = f_i \). The \( f_i' \) and \( U_i' \) are then the desired homeomorphisms and open sets.

**Theorem 3.** If \( E \) is infinite-dimensional and if \( f \in SH(E) \) such that \( f(B_1) \) is bounded, then there exists \( N > 0 \) and \( g \in H(E) \) such that \( g|B_1 = f|B_1 \) and \( g|(E-B_N) = \text{identity} \).

**Proof.** By Lemma 3 and its addendum, there exists, for \( i = 1, \cdots, n, f_i \in H(E) \) and open \( U_i \subset E - f_{i-1} \cdots f_0(B_1) \) \((f_0 = \text{identity})\) such that \( f_i|U_i = \text{identity} \), \( f = f_1 \cdots f_n \), and for \( i = 1, \cdots, n - 1, f_i \cdots f_0(B_1) \subset f_{i+1} \cdots f_1(B_1) \). Without loss of generality it can be assumed that each \( U_i \) is bounded. Choose \( N > 0 \) such that \( B_1 \cup f(B_1) \cup [U_{i-1}' U_i] \subset B_N \). Choose \( B_\gamma(x) \subset f_1(B_1) \) and \( B_\delta(y) \subset U_1 \). Let \( g' \in H(E) \) be such that \( g'[B_\gamma(x)] = B_1 \) and \( g'| [B_\delta(y) \cup (E-B_N)] = \text{identity} \). Redefine \( f_1 \) to be \( f_1 g' \). For each \( i = 1, \cdots, n \), by the Engulfing Theorem there exists \( g_i \in H(E) \) such that \( E - B_N \subset g_i(U_i) \) and \( g_i|f_i \cdots f_1[B_\gamma(x)] = \text{identity} \). Define \( f_i' = g f_i g_i^{-1} \). Then define the desired \( g \) by \( g = f_1' \cdots f_1(g')^{-1} \).

**Theorem 4.** \( S_E \) implies \( A_E \).

**Proof (For \( E \) infinite-dimensional).** Let \( C \) be a tame cell contained in \( \text{Int } B_1 \). By the Half-open Annulus Theorem [7], there exists a homeomorphism \( f \) of \( \text{Int } B_1 \) onto itself such that \( f(B_1/2) = C \). By a slight modification of Lemma 3.1 in [7], there exists a cell \( C' \subset B_1 \) such that \( C \cup B_1/2 \subset C' \) and \( B_1 - \text{Int } C' \) is a collar of \( C' \). Define \( g \in H(\text{Int } B_1) \rightarrow E \) radially in the following manner. Set \( S' = \text{Bd } C' \). For \( x \in E \), because of the construction of \( C' \) in Lemma 3.1 in [7], \( \text{Ray } [\theta: x] \cap S' \) is a single point—call it \( x_1 \). Also \( \text{Ray } [\theta: x] \cap S_1 = x_1 \). Let \( g \) map \( [x_1: x] \) linearly onto \([x_1: \infty]) \) and let \( g|C' = \text{identity} \). Set \( \tilde{f} = gfg^{-1} \). Then since \( S_E \) is true, \( \tilde{f} \in SH(E) \). Hence by Theorem 3 there exist \( N > 0 \) and \( \tilde{g} \in H(E) \) such that \( \tilde{g}|B_1 = \tilde{f}|B_1 \) and \( \tilde{g}|(E-B_N) = \text{identity} \). Let \( h \colon \text{Int } B_1 \rightarrow \text{Int } B_1 \) be defined by \( h = g^{-1}\tilde{g}g \). Finally \( h \) can be extended to the identity on \( S_1 \).

**Corollary 1.** \( A_E \) is true in all separable infinite-dimensional Fréchet spaces.

The annulus conjecture for the Hilbert cube has been observed to be true by Anderson in Corollary 10.6 of [2]. His proof of this also
follows from the fact that homeomorphisms of the Hilbert cube onto itself are stable, which he shows in that paper.

Let $HI(E)$ denote the homeomorphisms on $E$ which are isotopic to the identity. $I_E$ will be the conjecture that $HI(E) = H(E)$ (use orientation preserving homeomorphisms for $E$ finite-dimensional).

The following theorem is well known and can be found for example proved for the special spaces of $E^n$, $s$, and $I^n$ in [3], [10], and [1], respectively.

**Theorem 5.** $S_E$ implies $I_E$.

**Proof.** Let $h \in SH(E)$.

**Case 1.** Assume that $h$ is fixed on $P_i$. Define

$$H_t(x) = \frac{(1 + t)}{(1 - t)}h\left[\frac{(1 - t)}{(1 + t)}x\right],$$

for $0 \leq t < 1$, and $H_t(x) = x$, for $t = 1$. Let $x \in E$ and $\gamma > 0$, and choose $\delta = \max\left[\frac{||x|| + \gamma - 1}{||x|| + \gamma + 1}, 0\right]$. Then for any $y \in B_\gamma(x)$ and $t \in (0, 1]$, $H_t(y) = y$. Thus $H_t$ is an isotopy from $h$ to the identity.

**Case 2.** $h = h_\gamma \cdots h_1$, where each $h_i$ is fixed on some $B_\gamma(x_i)$. Then as in the above case define isotopies $H_t^i$ from $h_i$ to the identity. Finally define the isotopy $H_t = H_\gamma^i \cdots H_1^i$ from $h$ to the identity.

**Lemma 4.** Let $A_E$ hold, let $C$ be a tame cell, and let $B_\gamma(x) \subset \text{Int } C$. Then there exists $g \in H(E)$ such that $g(C) = B_{\gamma+1}(x)$ and $g|_{B_\gamma(x)} = \text{identity}$.

**Proof.** Let $h \in H(E)$ be such that $h(B_1) = C$. By $A_E$, let $f \in H(E)$ be such that $f(B_{1/2}) = h^{-1}(B_\gamma(x))$ and $f|_{E - B_1} = \text{identity}$. Define $j \in H(E)$ so that $j(B_{1/2}) = B_\gamma(x)$ and $j(B_1) = B_{\gamma+1}(x)$. Let $p_\gamma^t$ be the projection of $E - \theta$ onto $S_\gamma(x)$ along the rays emanating from $x$. Define $k \in H(E)$ so that $k|_{B_\gamma(x)} = h(j^{-1}|_{B_\gamma(x)})$, and for $y \in S_\gamma$, $r > \gamma$, $k(y) = p_\gamma^t k p_\gamma^t(y)$. Then set $g = k j f^{-1} h^{-1}$.

**Theorem 6.** $A_E$ and $I_E$ together imply $S_E$ for all normed linear spaces $E$ which contain a hyperplane homeomorphic to $E$.

**Proof.** Let $h \in H(E)$, and let $B_\gamma(x) \subset h(\text{Int } B_1)$. Then by Lemma 4, there exists $g \in H(E)$ such that $gh(B_1) = B_{\gamma+1}(x)$ and $g|_{B_\gamma(x)} = \text{identity}$. Let $k \in H(E)$ be such that $k(B_{\gamma+1}(x)) = B_1$ and $k|_{E - B_N} = \text{identity}$ for some $N$. In [6] it is seen that all hyperplanes of $E$ are homeomorphic to $S_1$ in $E$. Then because $I_E$ holds, there exists an isotopy $F_t : S_1 \rightarrow S_1$, $0 \leq t \leq 1$, such that $F_0 = kgh|_{S_1}$ and $F_1 = \text{identity}$. Let $p_\gamma$, $r > 0$, be the radial projection of $E - \theta$ onto $S_r$. Then define $f \in H(E)$ as follows. For $x \in S_r$, $1 \leq r \leq 2$, let $f(x) = p_\gamma F_{r-1} p_1(x)$, $f|_{B_1}$...
$= kgh|B_1$, and $f| (E - B_2) = \text{identity}$. Then $h = g^{-1}k^{-1}f(f^{-1}kgh)$, which is stable.

**Corollary 2.** $A_E$ and $I_E$ together are equivalent to $S_E$ for all normed linear spaces $E$ which contain a hyperplane homeomorphic to $E$.

Similar arguments give the finite-dimensional result found in [3].

**Theorem 7.** $A_n$ and $I_{n-1}$ together are equivalent to $S_n$.

**References**