ANNULUS CONJECTURE AND STABILITY OF HOMEOMORPHISMS IN INFINITE-DIMENSIONAL NORMED LINEAR SPACES

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Abstract. If $E$ is an arbitrary infinite-dimensional normed linear space, it is shown that if all homeomorphisms of $E$ onto itself are stable, then the annulus conjecture is true for $E$. As a result, this confirms that the annulus conjecture for Hilbert space is true. A partial converse is that for those spaces $E$ which have some hyperplane homeomorphic to $E$, if the annulus conjecture is true for $E$ and if all homeomorphisms of $E$ onto itself are isotopic to the identity, then all homeomorphisms of $E$ onto itself are stable.

Let $E$ denote a normed linear space with zero element $0$, and let $B_r(x)$ be the ball in $E$ of radius $r$ centered at $x$, $S_r(x) = \text{Bd } B_r(x)$, $B_r = B_r(0)$, and $S_r = S_r(0)$. $H(E)$ and $SH(E)$ will denote the homeomorphisms on $E$ and the stable homeomorphisms on $E$, respectively.

From [5] and [6] there is an inversion homeomorphism $\iota\in H(E)$ for $E$ infinite-dimensional such that $\iota(B_1) = E - \text{Int } B_1$, $\iota| S_1 = \text{identity}$, and $\iota^{-1} = \iota$. It can be seen in [7] and [8] that this homeomorphism is a useful tool in proving topological properties of infinite-dimensional normed linear spaces. In this paper $\iota$ will be used in establishing a type of engulfing theorem for infinite-dimensional $E$ which is somewhat analogous to Stallings’ Engulfing Theorem in [9]. Brown and Gluck in [3] showed that $SH(E) = H^+(E)$ implies the annulus conjecture for finite-dimensional $E$, where $H^+(E)$ is the group of orientation preserving homeomorphisms of $E$. As an application of the engulfing theorem in this paper, it will be shown that $SH(E) = H(E)$ implies the annulus conjecture for infinite-dimensional $E$ (a slightly different technique than used in [3]). As a result, since Wong in [10] showed that $SH(l_2) = H(l_2)$, the annulus conjecture is true in Hilbert space and is consequently true in all infinite-dimensional separable Banach spaces, or more generally, in all infinite-dimensional separable Fréchet spaces (see [4] and [1]).

By a cell in $E$ is meant a subset $C$ of $E$ such that there exists a homeomorphism from the pair $(B_1, S_1)$ onto the pair $(C, \text{Bd } C)$. A
closed subset $K$ of $E$ is a collar of $C$ if there exists a homeomorphism $h$ from the pair $(B_2, B_1)$ onto the pair $(K \cup C, C)$ such that $h(S_2) = \text{Bd} \ (K \cup C)$. $C$ is tame if there is an $h \in H(E)$ such that $h(B_1) = C$. In [7] and [8] it is shown that $C$ is tame if and only if it has a collar. By a sphere in $E$ is meant a closed homeomorph of $S_1$ in $E$. A sphere is tame if it is the boundary of a tame cell. Sanderson showed in [8] that a sphere is tame if and only if it is bicollared. $A \subset E$ is an annulus if there exists a homeomorphism $h$ from $B_2 - \text{Int} \ B_1$ onto $A$ such that $\text{Bd} \ A = h(S_1 \cup S_2)$.

The annulus conjecture for $E$ is stated in [7] as follows.

$A_E$: If $C$ is a tame cell contained in $\text{Int} \ B_1$, then there exists a homeomorphism $h$ of $B_1$ onto itself such that $h(B_{1/2}) = C$ and $h|S_1 = \text{identity}$.

For infinite-dimensional $E$, it can be seen by using $i$ that $A_E$ is equivalent to the statement: the region between two disjoint tame spheres is an annulus.

$h \in H(E)$ is stable if it can be written as a finite composition of elements of $H(E)$ each of which is the identity on some open subset of $E$. The following is the stability conjecture for $E$.

$S_E$: $SH(E) = H(E)$ (for $E$ finite-dimensional $SH(E) = H^+(E)$).

**Lemma 1.** If $E$ is infinite-dimensional and $C$ is a collared cell with collar $K$, then there exists a collar $K' \subset K$ of $C$ and $g \in H(E)$ such that $g(C) = E - \text{Int} \ B_1$ and $g(C \cup K') = E - \text{Int} \ B_{1/2}$.

**Proof.** Let $h: (C \cup K, \text{Bd} \ [C \cup K]) \rightarrow (B_2, S_3)$ be a homeomorphism such that $h(C) = B_1$. Set $K' = h^{-1}(B_2 - \text{Int} \ B_1)$. By Lemma 6.1 of [7], there exists $f \in H(E)$ such that $f|K' \cup C = h|K' \cup C$. Let $k: B_2 - \text{Int} \ B_1 \rightarrow B_1 - \text{Int} \ B_{1/2}$ be a homeomorphism such that $k(S_2) = S_{1/2}$ and $k|S_1 = \text{identity}$. Define the homeomorphism $j': E - \text{Int} \ B_1 \rightarrow B_1 - \theta$ by letting $j'(x) = k(x)$ if $x \in E - B_2$, and, for $E - B_2$, letting $j'$ map $[x: \infty)$, $x \in B_2$, linearly onto $[k(x): \theta)$. Let $j: B_1 - \theta \rightarrow B_1$ be a homeomorphism such that $j|\ (B_1 - \text{Int} \ B_{1/2}) = \text{identity}$. Define $g' \in H(E)$ by $g'(x) = i(x)$ if $x \in B_1$, and $g'(x) = jf'(x)$ if $x \in E - B_1$. Finally define $g \in H(E)$ by $g = g'f$.

**Theorem 1 (Engulfing theorem).** Let $E$ be infinite-dimensional, let $A \subset E$, let $C$ be a collared cell in $E$ with collar $K$ such that $(C \cup K) \cap A = \emptyset$, and let $U \subset E$ be open such that $[E - (C \cup K)] \cap U \neq \emptyset$. Then there exists $h \in H(E)$ such that $A \subset h(U)$ and $h|C = \text{identity}$.

**Proof.** By Lemma 1, let $K' \subset K$ be a collar of $C$ and $g \in H(E)$ be such that $g(C) = E - \text{Int} \ B_1$ and $g(C \cup K') = E - \text{Int} \ B_{1/2}$. Choose $B_\gamma(x) \subset g(U) \cap \text{Int} \ B_{1/2}$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_\gamma(x)]$ and
Theorem 2. Let $E$ be infinite-dimensional, let $A \subset E$, and let $C$ be a collared cell in $E$ with collar $K$ such that $C \cup \text{Cl} A \neq E$. Then there exists $h \in H(E)$ such that $A \subset h(C \cup K)$ and $h|C = \text{identity}$.

Proof. By Lemma 1, let $K' \subset K$ be a collar of $C$ and $g \in H(E)$ be such that $g(C) = E - \text{Int} B_1$ and $g(C \cup K') = E - \text{Int} B_{1/2}$. Choose $B_r(x) \subset \text{Int} B_1 - g(A)$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_r(x)]$ and $f|E - B_1 = \text{identity}$. Define $h \in H(E)$ by $h = g^{-1}f^{-1}g$.

Lemma 2. Let $A$ be a nondense subset of $E$, and let $f \in H(E)$ and $U \subset A$ be open such that $f|U = \text{identity}$. Then there exist $f_1, f_2, f_3 \in H(E)$ and $U_1, U_2, U_3$ open subsets of $E$ such that $U_1 \subset E - A$, $U_2 \subset E - f_1(A)$, $U_3 \subset E - f_2f_1(A)$, $f = f_3f_2f_1$, and $f|U_i = \text{identity}$ for $i = 1, 2, 3$.

Proof. Let $B_r(x) \subset U$ and $B_s(y) \subset E - A$. Let $N > M > 0$ be such that $\text{Int} B_s(y) \cap \text{Int} B_M(x) \neq \emptyset$ and $\text{Int} B_s(y) \cap [E - B_N(x)] \neq \emptyset$. Define $f_1 \in H(E)$ such that $f_1|B_M(x) = B_r(x)$ and $f_1|E - B_N(x) = \text{identity}$. Define $f_2, f_3 \in H(E)$ by $f_2 = f$ and $f_3(x) = f^{-1}f^{-1}(x)$ if $x \in f[B_N(x)]$, and $f_3(x) = x$ if $x \in E - f[B_N(x)]$. Choose $U_1$ open in $B_r(y) - B_N(x)$, $U_2$ open in $B_r(x) \cap f_1[B_s(y)]$, and $U_3$ open in $f[B_s(y) - B_N(x)]$.

Lemma 3. If $f \in \mathcal{H}(E)$, then there exist, for $i = 1, \ldots, n$, $f_i \in H(E)$ and open $U_i \subset E - f_{i-1} \cdots f_0(B_1)$ ($f_0 = \text{identity}$) such that $f_i|U_i = \text{identity}$ and $f = f_n \cdots f_1$.

Proof. Let $f = g_m \cdots g_1$ and $V_i, i = 1, \ldots, m$, be open such that $g_i|V_i = \text{identity}$. Let $k$ be such that $1 \leq k \leq m$. If $V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)] \neq \emptyset$ ($g_0 = \text{identity}$), then choose $U_{k1} \subset V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)]$ and let $f_{k1} = g_k$. If $V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)] = \emptyset$, then by Lemma 2, there exist $f_{k1}, f_{k2}, f_{k3} \in H(E)$, open $U_{k1} \subset E - g_{k-1} \cdots g_3g_0(B_1)$, open $U_{k2} \subset E - f_{k1}g_{k-1} \cdots g_{1g_0}(B_1)$, and open $U_{k3} \subset E - f_{k2}f_{k1}g_{k-1} \cdots g_{1g_0}(B_1)$ such that $g_k = f_{k3}f_{k2}f_{k1}$ and $f_{k1}|U_{ki} = \text{identity}$, $i = 1, 2, 3$. The $f_{ij}$ and $U_{ij}$ can be relabeled to give the desired $f_i$ and $U_i, i = 1, \ldots, n$.

Addendum to Lemma 3. In Lemma 3 the $f_i$ can further be chosen so that $f_i \cdots f_1(B_1) \subset f_{i+1} \cdots f_1(B_1)$ for $i = 1, \ldots, n - 1$.

Proof. Let $U'_n = U_n$ and $f'_n = f_n$. For $k$ such that $1 \leq k \leq n - 1$, define $U'_{n-k}, f'_{n-k}, U''_{n-k+1},$ and $f''_{n-k+1}$ by an inductive step as follows.
Choose \( B_\gamma(x) \subseteq f_i^{-1} \cdots f_{n-k}^{-1} f_{n-k+1} \cdots f_1(B_1) \), \( B_\delta(y) \subseteq f_i^{-1} \cdots f_{n-k}^{-1} (U_{n-k+1}) \) such that \( B_\gamma(x) \cap (B_\delta(y) \cup B_\epsilon(z)) = \emptyset \). Let \( g \in H(E) \) be such that \( g(B_1) = B_\gamma(x) \) and \( g| (B_\delta(y) \cup B_\epsilon(z)) = \text{identity} \). Define \( U_{n-k} = f_{n-k} \cdots f_1(\text{Int } B_\delta(y)) \), \( f_{n-k} = f_{n-k} \cdots f_1(g_i^{-1} \cdots f_{n-k+1} f_{n-k}^{-1}) \), \( U_{n-k+1} = f_{n-k} \cdots f_1(\text{Int } B_\epsilon(z)) \), and \( f_{n-k+1} = f_{n-k+1} f_{n-k} (f_{n-k}^{-1})^{-1} \). Finally let \( U_i'' = U_i' \) and \( f_i'' = f_i' \). The \( f_i'' \) and \( U_i'' \) are then the desired homeomorphisms and open sets.

**Theorem 3.** If \( E \) is infinite-dimensional and if \( f \in SH(E) \) such that \( f(B_1) \) is bounded, then there exists \( N > 0 \) and \( g \in H(E) \) such that \( g| B_1 = f| B_1 \) and \( g| (E - B_N) = \text{identity} \).

**Proof.** By Lemma 3 and its addendum, there exists, for \( i = 1, \ldots, n \), \( f_i \in H(E) \) and open \( U_i \subseteq E - f_i \cdots f_0(B_1) \) \( (f_0 = \text{identity}) \) such that \( f_i| U_i = \text{identity}, \ f = f_n \cdots f_1 \), and for \( i = 1, \ldots, n-1 \), \( f_i \cdots f_0(B_1) \subseteq f_{i+1} \cdots f_1(B_1) \). Without loss of generality it can be assumed that each \( U_i \) is bounded. Choose \( N > 0 \) such that \( B_1 \cup f(B_1) \cup [U_{n-1} U_1] \subseteq B_N \). Choose \( B_\gamma(x) \subseteq f_1(B_1) \) and \( B_\delta(y) \subseteq U_1 \). Let \( g' \in H(E) \) be such that \( g'[B_\gamma(x)] = B_1 \) and \( g'| (B_\delta(y) \cup (E - B_N)) = \text{identity} \). Redefine \( f_1 \) to be \( f_1 g' \). For each \( i = 1, \ldots, n \), by the Engulfing Theorem there exists \( g_i \in H(E) \) such that \( E - B_N \subseteq g_i(U_i) \) and \( g_i| f_i \cdots f_1[B_\gamma(x)] = \text{identity} \). Define \( f_i' = g f_i g_i^{-1} \). Then define the desired \( g \) by \( g = f_n' \cdots f_1'(g')^{-1} \).

**Theorem 4.** \( S_E \) implies \( A_E \).

**Proof (For \( E \) infinite-dimensional).** Let \( C \) be a tame cell contained in \( \text{Int } B_1 \). By the Half-open Annulus Theorem [7], there exists a homeomorphism \( f \) of \( \text{Int } B_1 \) onto itself such that \( f(B_{1/2}) = C \). By a slight modification of Lemma 3.1 in [7], there exists a cell \( C' \subseteq B_1 \) such that \( C \cup B_{1/2} \subseteq C' \) and \( B_1 - \text{Int } C' \) is a collar of \( C' \). Define \( g : \text{Int } B_1 \to E \) radially in the following manner. Set \( S' = \partial C' \). For \( x \in E \), because of the construction of \( C' \) in Lemma 3.1 in [7], \( (\text{Ray } [\theta : x]) \cap S' \) is a single point—call it \( x_1 \). Also \( (\text{Ray } [\theta : x]) \cap S_1 = x_1 \). Let \( g \) map \( [x_1 : x] \) linearly onto \( [x_1 : \infty] \) and let \( g| C = \text{identity} \). Set \( f = g f g^{-1} \). Then since \( S_E \) is true, \( f \in SH(E) \). Hence by Theorem 3 there exist \( N > 0 \) and \( \tilde{g} \in H(E) \) such that \( \tilde{g}| B_1 = f| B_1 \) and \( \tilde{g}| (E - B_N) = \text{identity} \). Let \( h : \text{Int } B_1 \to \text{Int } B_1 \) be defined by \( h = g^{-1} \tilde{g} g \). Finally \( h \) can be extended to the identity on \( S_1 \).

**Corollary 1.** \( A_E \) is true in all separable infinite-dimensional Fréchet spaces.

The annulus conjecture for the Hilbert cube has been observed to be true by Anderson in Corollary 10.6 of [2]. His proof of this also...
follows from the fact that homeomorphisms of the Hilbert cube onto itself are stable, which he shows in that paper.

Let \( HIE \) denote the homeomorphisms on \( E \) which are isotopic to the identity. \( IB \) will be the conjecture that \( HIE = H(E) \) (use orientation preserving homeomorphisms for \( E \) finite-dimensional).

The following theorem is well known and can be found for example proved for the special spaces of \( E^n \), \( s \), and \( I^\infty \) in [3], [10], and [1], respectively.

**Theorem 5.** \( SE \) implies \( IE \).

**Proof.** Let \( h \in SH(E) \).

Case 1. Assume that \( h \) is fixed on \( P_1 \). Define

\[
H_t(x) = \frac{(1 + t)}{(1 - t)}h[(1 - t)/(1 + t)x],
\]

for \( 0 \leq t < 1 \), and \( H_t(x) = x \), for \( t = 1 \). Let \( x \in E \) and \( \gamma > 0 \), and choose

\[
\delta = \max \left\{ \frac{\|x\| + \gamma - 1}{\|x\| + \gamma + 1} \right\},
\]

Then for any \( y \in B_\gamma(x) \) and \( t \in (0, 1) \), \( H_t(y) = y \). Thus \( H_t \) is an isotopy from \( h \) to the identity.

Case 2. \( h = h_n \cdots h_1 \), where each \( h_i \) is fixed on some \( B_\gamma_i(x_i) \). Then as in the above case define isotopies \( H_t^i \) from \( h_i \) to the identity. Finally define the isotopy \( P = P_n \cdots P_1 \) from \( h \) to the identity.

**Lemma 4.** Let \( AE \) hold, let \( C \) be a tame cell, and let \( B_\gamma(x) \subset Int C \). Then there exists \( g \in H(E) \) such that \( g(C) = B_{\gamma+1}(x) \) and \( g|B_\gamma(x) = \text{identity} \).

**Proof.** Let \( h \in H(E) \) be such that \( h(B_1) = C \). By \( AE \), let \( f \in H(E) \) be such that \( f(B_{1/2}) = h^{-1}(B_{\gamma}(x)) \) and \( f(E - B_1) = \text{identity} \). Define \( j \in H(E) \) so that \( j(B_{1/2}) = B_\gamma(x) \) and \( j(B_1) = B_{\gamma+1}(x) \). Let \( p_\gamma^x \) be the projection of \( E - \theta \) onto \( S_\gamma(x) \) along the rays emanating from \( x \). Define \( k \in H(E) \) so that \( k|B_\gamma(x) = hf_j^{-1}|B_\gamma(x) \), and for \( y \in S_\gamma, r > \gamma, k(y) = p_\gamma^x k p_\gamma^x(y) \). Then set \( g = k f_j^{-1} h^{-1} \).

**Theorem 6.** \( AE \) and \( IE \) together imply \( SE \) for all normed linear spaces \( E \) which contain a hyperplane homeomorphic to \( E \).

**Proof.** Let \( h \in H(E) \), and let \( B_\gamma(x) \subset h(\text{Int } B_1) \). Then by Lemma 4, there exists \( g \in H(E) \) such that \( gh(B_1) = B_{\gamma+1}(x) \) and \( g|B_\gamma(x) = \text{identity} \). Let \( k \in H(E) \) be such that \( k(B_{\gamma+1}(x)) = B_1 \) and \( k|E - B_N = \text{identity for some } N \). In [6] it is seen that all hyperplanes of \( E \) are homeomorphic to \( S_1 \) in \( E \). Then because \( IE \) holds, there exists an isotopy \( F_t: S_1 \to S_1 \), \( 0 \leq t \leq 1 \), such that \( F_0 = kgh \) and \( F_1 = \text{identity} \). Let \( p_r, r > 0 \), be the radial projection of \( E - \theta \) onto \( S_r \). Then define \( f \in H(E) \) as follows. For \( x \in S_r, 1 \leq r \leq 2 \), let \( f(x) = p_rF_{r-1}p_1(x), f|B_1 \)
= kgfh | B₁, and \( f| (E-B₂) = \text{identity} \). Then \( h = g^{-1}k^{-1}f(f^{-1}kgfh) \), which is stable.

**Corollary 2.** \( A_E \) and \( I_E \) together are equivalent to \( S_E \) for all normed linear spaces \( E \) which contain a hyperplane homeomorphic to \( E \).

Similar arguments give the finite-dimensional result found in [3].

**Theorem 7.** \( A_n \) and \( I_{n-1} \) together are equivalent to \( S_n \).

**References**


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