COMPONENT FUNCTORS

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Introduction. Many concrete categories $\mathcal{C}$ admit, besides the usual forgetful functor, another such functor $V: \mathcal{C}\to \mathcal{S}$ (where $\mathcal{S}$ is the category of sets) which may reasonably be termed a "component functor." For example, let $M$ be a monoid and let $\mathcal{C}$ be the category of sets on which $M$ acts on the left. If $M$ acts on $A$, let $\sim$ be the equivalence relation generated by $a\sim ma$ for $a\in A, m\in M$. Then the component functor $V$ takes $A$ to the set $A/\sim$ of connected components of $A$ under the action of $M$.

Many such functors are especially interesting in that they have right adjoints $U: \mathcal{S}\to \mathcal{C}$ which are monadic (i.e., triplable). In this paper we show that this situation will arise whenever $\mathcal{C}$ is comonadic over $\mathcal{S}$, and exhibit some typical examples.

1. Statement of the theorem. Let $F=(F, e, \delta)$ be a comonad over $\mathcal{S}$. We recall that this means that $F: \mathcal{S}\to \mathcal{S}$ is a functor and that $e: F\to 1_{\mathcal{S}}$ and $\delta: F\to F^2 = F\circ F$ are natural transformations such that $F\delta \circ \delta = \delta F \circ \delta$ and $F\delta \circ \delta = \delta F \circ \delta$. An algebra over $F$ is a pair $(A, \xi)$ consisting of a set $A$ and a map $\xi: A\to FA$ such that $\xi A \circ \xi = 1_A$ and $F(\xi) \circ \xi = \delta A \circ \xi$. A homomorphism $f: (A, \xi)\to (B, \rho)$ is a map $f: A\to B$ such that $\rho \circ f = F(f) \circ \xi$. Let $\mathcal{C}$ be the category of algebras over $F$.

We note that $\mathcal{C}$ possesses coproducts, which are formed by disjoint union just as in the category of sets. For example, the coproduct of $(A, \xi)$ and $(B, \rho)$ is $(A \coprod B, \omega)$ where the map $\omega: A \coprod B\to F(A \coprod B)$ is the composition

$$A \coprod B \xrightarrow{\xi \coprod \rho} F(A) \coprod F(B) \xrightarrow{\gamma} F(A \coprod B)$$

and $\gamma$ is defined by $\gamma \circ i_{F(A)} = F(i_A)$, $\gamma \circ i_{F(B)} = F(i_B)$. If $X$ is any set, we shall denote by $X\cdot A$ the coproduct of $|X|$ copies of $A$.

Note also that $\mathcal{C}$ has a terminal object $T$. Using 1 to denote a fixed one-element set, we have $T = F(1)$ equipped with the map $\delta_1: T\to F(T)$. Now we are ready to state the

Theorem. The functor $U: \mathcal{S}\to \mathcal{C}$ defined by $U(X) = X\cdot T$ is monadic.

The left adjoint $V$ to $U$ is called the component functor associated with $\mathcal{C}$.

Received by the editors March 27, 1969.
2. Proof of the theorem. First we shall prove that $U$ has a left adjoint $V$, by appealing to the adjoint-functor theorem of Freyd [4]. Observe that a map $f: (A, \xi) \to X \cdot T$ is a homomorphism iff the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \cdot T \\
\downarrow \xi & & \downarrow X \cdot \delta_1 \\
F(A) & \xrightarrow{F(f)} & F(X \cdot T)
\end{array}
\]

commutes. There is a unique homomorphism $e_A: A \to T$ which is equal to $F(\varepsilon) \circ \xi: A \to F(A) \to F(1) = T$ where $\varepsilon: A \to 1$ is the unique set map. Then for each $a \in A, f(a)$ must be the element $e_A(a)$ in some summand of $X \cdot T$ labeled by $\chi(a) \in X$. In this way we obtain a function $\chi: A \to X$ and by chasing elements in the diagram it is easily seen that the diagram commutes iff for every $a \in A$, we have the additional property that

$$F(f)\xi(a) = F(i_{\chi(a)})\delta_1 e_A(a).$$

Using this characterization of homomorphisms $A \to X \cdot T$ we shall show that the functor $U$ preserves limits.

To show that $U$ preserves equalizers, let $f, g: X \to Y$ and let $h: E \to X$ be $eq(f, g)$. We claim that $h: E \to X$ is the equalizer in $\mathcal{A}$ of $f \cdot T$ and $g \cdot T$. For, suppose that $k: (A, \xi) \to X \cdot T$ is such that $(f \cdot T) \circ k = (g \cdot T) \circ k$. Then there is a unique set map $j: A \to E \cdot T$ such that $k = (h \cdot T) \circ j$, and we need only show that $j$ is a homomorphism. Let $k(a)$ be the element $e_A(a)$ in the summand of $X \cdot T$ labeled by $\chi(a)$ and let $j(a)$ be the element $e_A(a)$ in the summand of $E \cdot T$ labeled by $\eta(a)$. Then

$$F(h \cdot T)F(j)\xi(a) = F(k)\xi(a)$$

$$= F(i_{\chi(a)})\delta_1 e_A(a)$$

$$= F(h \cdot T)F(i_{\eta(a)})\delta_1 e_A(a).$$

Since $F(h \cdot T)$ is monic, $F(j)\xi(a) = F(i_{\eta(a)})\delta_1 e_A(a)$ and thus $j$ is indeed a homomorphism.

Next, it is rather more difficult to show that $U$ preserves products; for simplicity we write the proof for binary products but the proof for infinitary products is similar. Thus we must prove that the product in $\mathcal{A}$ of $X \cdot T$ and $Y \cdot T$ is $(X \times Y) \cdot T$ where $X \times Y$ is the product.
in S. Suppose $f: (A, \xi) \to X \cdot T$ and $g: (A, \xi) \to Y \cdot T$, where $f(a)$ is the $e_A(a)$ in the $\chi(a)$-summand and $g(a)$ is the $e_A(a)$ in the $\eta(a)$-summand. Define $h: A \to (X \times Y) \cdot T$ by requiring $h(a)$ to be $e_A(a)$ in the summand labeled by $(\chi(a), \eta(a))$. Then clearly we need only show that $h$ is a homomorphism.

First, we note that

$$F(f)\xi(a) = F(i_{\chi(a)})\delta_1e_A(a)$$

$$= F(i_{\chi(a)})\delta_1F(\xi)\xi(a)$$

$$= F(i_{\chi(a)})F^2(\xi)\delta_A\xi(a)$$

$$= F(i_{\chi(a)}) \circ F(\xi) \circ \xi(a)$$

$$= F(i_{\chi(a)}) \circ e_A\xi(a),$$

and similarly $F(g)\xi(a) = F(i_{\eta(a)}) \circ e_A\xi(a)$. To show that $h$ is a homomorphism, i.e., to show that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & (X \times Y) \cdot T \\
\downarrow \xi & & \downarrow (X \times Y) \cdot \delta_1 \\
F(A) & \xrightarrow{F(h)} & F((X \times Y) \cdot T)
\end{array}
\]

commutes, we must show that for all $a \in A$,

$$F(h)\xi(a) = F(i_{\chi(a), \eta(a)}) \circ e_A\xi(a).$$

The result would be immediate if we knew that the map $(F(p_1), F(p_2)): F((X \times Y) \cdot T) \to F(X \cdot T) \times F(Y \cdot T)$ is one-to-one. Unfortunately, there are many examples of comonads for which it is not one-to-one. However, it will suffice to show that $(F(p_1), F(p_2))$ is one-to-one on the image of $\gamma \circ (X \times Y) \cdot \delta_1$. In fact, we shall show that the composition

$$k = (F(p_1), F(p_2)) \circ \gamma \circ (X \times Y) \cdot \delta_1: (X \times Y) \cdot T \to F(X \cdot T) \times F(Y \cdot T)$$

is actually a one-to-one function.

A little computation using the definition of $\gamma$ shows that $k = (\gamma_X, \gamma_Y) \circ (X \times Y) \cdot \delta_1$, where $\gamma_X: (X \times Y) \cdot F(T) \to F(X \cdot T)$ is defined by $\gamma_X \circ i_{(x, y)} = F(i_z) \circ F(T) \to F(X \cdot T)$ and likewise $\gamma_Y \circ i_{(x, y)} = F(i_y)$. Then we have $k \circ i_{(x, y)} = (F(i_z), F(i_y)) \circ \delta_1: T \to F(T) \to F(X \cdot T) \times F(Y \cdot T)$. But $\delta_1$, $F(i_z)$, and $F(i_y)$ are one-to-one and hence $k \circ i_{(x, y)}$ is one-to-one for every $(x, y) \in X \times Y$. 

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To show that \( k \) is one-to-one, it now suffices to show that the images of distinct summands under \( k \) are disjoint; that is, if \((x, y) \neq (\hat{x}, \hat{y})\) then the maps \((F(i_x), F(i_y)) \circ \delta_1 \) and \((F(i_{\hat{x}}), F(i_{\hat{y}})) \circ \delta_1 : T \to F(X \cdot T) \times F(Y \cdot T)\) have disjoint images. In fact, more is true: if \( x \neq \hat{x} \) then the images of \( F(i_x) \circ \delta_1 \) and \( F(i_{\hat{x}}) \circ \delta_1 \) are disjoint (and similarly if \( y \neq \hat{y} \)). For, if they were not disjoint, they would continue to fail to be disjoint upon application of the map \( \varepsilon_{X \cdot T} \). But the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\delta_1} & F(T) \\
1_T & \downarrow & \downarrow \varepsilon_{T} \\
T & \xrightarrow{i_x} & X \cdot T
\end{array}
\]

shows that this would be impossible. Hence \( k \) is one-to-one, and we have completed the proof that \( U \) preserves limits.

Now, in order to show that \( U \) has a left adjoint \( V \), it suffices to show that for each \((A, \xi)\) there is a family \( \{B_\alpha\} \) of sets such that whenever \( f : (A, \xi) \to X \cdot T \), there is a homomorphism \( \eta : (A, \xi) \to B_\alpha \cdot T \) and a set map \( g : B_\alpha \to X \) such that \( f = (g \cdot T) \circ \eta \). But given \( f \), its image contains at most \(|A|\) elements and thus is contained in \( B \cdot T \) where \( B \subseteq X \) and \(|B| \leq |A| \). Thus, for a solution set for \((A, \xi)\) we can take the set of all subsets of \( A \). This establishes the existence of the component functor \( V : \mathcal{C} \to \mathcal{S} \), by Freyd's adjoint functor theorem.

Finally we proceed to show that \( U \) is monadic by appealing to Beck's Triplability Theorem \([1], [3]\). Let \( d_0, d_1 \). \( X \to Y \) and let \( q : Y \to W \) be \( \text{coeq}(d_0, d_1) \). Suppose that in \( \mathcal{C} \) we have a diagram

\[
\begin{array}{ccc}
X \cdot T & \xrightarrow{d_0 \cdot T} & Y \cdot T \\
\downarrow z & \Rightarrow & \downarrow s \\
Z, & \xrightarrow{t} & \end{array}
\]

where \( z = \text{id} \), \( z = (d_1 \cdot T) \circ t \), \( (d_0 \cdot T) \circ t = \text{id} \), and \( z \circ (d_0 \cdot T) = z \circ (d_1 \cdot T) \). Then \( z = \text{coeq}(d_0 \cdot T, d_1 \cdot T) \), and in this situation Beck's theorem requires that \( Z \cong W \cdot T \), \( z \cong q \cdot T \). These equations are certainly true set-wise, so what we have to show is that the \( F \)-algebra structure on \( Z \) is that of \( W \cdot T \). But this is easily seen by observing that \( W \cdot T \) does function as the coequalizer of \( d_0 \cdot T \) and \( d_1 \cdot T \) in \( \mathcal{C} \). This completes the proof of our theorem.

3. **Examples.** A. Let \( M \) be a monoid, let \( T \) be a set on which \( M \) acts, and let \( \text{Mac}(M, T) \) be the category whose objects are sets \( A \) on
which $M$ acts, equipped with $M$-homomorphisms $e_A: A \to T$. A map $f: (A, e_A) \to (B, e_B)$ is to be an $M$-homomorphism such that $e_B \circ f = e_A$. Mac($M$, $T$) is comonadic over $S$ by [2]. Then $U(X) = (X \times T, e_{X \times T})$ where $m(x, t) = (x, ml)$ and $e_{X \times T} = p_2$. The component functor takes $(A, e_A)$ to the set of connected components of $A$ under the action of $M$. (This example was pointed out to the author by John Isbell in a personal communication.)

B. Let $\alpha$ be the category of sets equipped with equivalence relations. Using $[a]$ to denote the equivalence class of $a$, define a map $f: (A, \sim) \to (B, \sim)$ to be a function $f: A \to B$ such that $f[a] = [fa]$ for all $a \in A$. Then $\alpha$ is comonadic, $U(X) = (X, =)$, and $V(A, \sim) = A/\sim$.

C. For a more elaborate example, construct a comonad $(F, \epsilon, \delta)$ as follows. Let $F(A) = A \times 2A$, $e_A = \rho_1$, and define $\delta_A$ by $\delta_A(a, \alpha) = (a, \alpha, \{(b, \alpha) \mid b \in \alpha\})$. Then it is easy to compute that an algebra over this monad is essentially a set $A$ equipped with a function $\phi_A: A \to 2^A$ with the property that $\phi_A(a) = \phi_A(a')$ whenever $a' \in \phi_A(a)$. A homomorphism $f: (A, \phi_A) \to (B, \phi_B)$ is a function such that $f(\phi_A(a)) = \phi_B(f(a))$ for all $a \in A$. The terminal object is $T = \{0, 1\}$ where $\phi_T(0) = \emptyset$ and $\phi_T(1) = \{1\}$. The component functor takes $(A, \phi_A)$ to the set

$$\{a \mid a \in A, \phi_A(a) = \emptyset\} \cup \{\phi_A(a) \mid a \in A, \phi_A(a) \neq \emptyset\}.$$

This case is particularly interesting since it admits another monadic functor $U: S \to \alpha$ where $U(A) = (A \sqcup \{1\}, \phi)$, $\phi(a) = \{a\}$, and $\phi(1) = \emptyset$. The left adjoint to this functor takes $(A, \phi_A)$ to the set $\{\phi_A(a) \mid a \in A, \phi_A(a) \neq \emptyset\}$. Thus, a category can admit reasonable "component functors" other than those for which we have provided a categorical framework.

**References**


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