LARGE DEVIATION PROBABILITIES FOR POSITIVE RANDOM VARIABLES

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1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of positive, independent and identically distributed random variables (r.v.'s) with distribution function (d.f.) $F$. Set $S_n = \sum_{k=1}^{n} X_k$. Following Heyde [2], we call $P(S_n > t_n)$ a large deviation probability when $t_n \to +\infty$ as $n \to +\infty$.

2. Results. The following theorem sharpens Heyde's result at the expense of a restriction to positive r.v.'s with slowly varying tail distributions.

Theorem. Let $\{t_n\}$ be a sequence of positive numbers with $t_n \to +\infty$ in such a way that $n(1 - F(t_n)) \to 0$ as $n \to +\infty$. If $1 - F$ is a slowly varying function, then

(1) $P(S_n > t_n) \sim nP(X_1 > t_n) \sim P(\max_{i \leq n} A_i > t_n)$, 

as $n \to +\infty$.

The right-hand side of (1) is a consequence of the following lemma. If $n(1 - F(t_n)) \to 0$ as $n \to +\infty$, then

(2) $nP(X_1 > t_n) \sim P(\max_{i \leq n} X_i > t_n)$, \hspace{1cm} (n \to +\infty).

Heyde, [2], obtains this result from Bonferroni's inequalities; it is also a simple consequence of the following inequality:

$n(1 - F(t_n)) \geq 1 - F^n(t_n) \geq 1 - e^{-n(1-F(t_n))}$.

Proof of the theorem. Let $M_n = \max_{i \leq n} X_i$. It is easy to see that

(3) $P(S_n > t_n) = P(M_n > t_n) + P(M_n \leq t_n, S_n > t_n)$.

Hence, using only the hypothesis of the lemma and (2), we find that

(4) $\liminf_n \{P(S_n > t_n)/nP(X_1 > t_n)\} \geq 1$.

In order to prove the other half of (1) we let $S_{an}$ be the sum of the truncates of $X_1, X_2, \cdots, X_n$ at the point $t_n$. Then, from equation (3),

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(5) \( P(S_n > t_n) \leq P(M_n > t_n) + P(S_{nn} > t_n) \).

Applying Markov's inequality to \( P(S_{nn} > t_n) \) we find that

(6) \( P(S_{nn} > t_n) \left/ \left\{ nP(X_1 > t_n) \right\} \right. \leq \int_0^{t_n} x dF(x) \left/ \left\{ t_n P(X_1 > t_n) \right\} \right. \).

Since \( 1 - F \) is a slowly varying function, we may use Karamata's theorem \([1, p. 273]\), to obtain the following limit.

\[
\int_0^{t_n} (1 - F(y)) dy / \{ t_n P(X_1 > t_n) \} \to 1,
\]

as \( n \to +\infty \). Then, after integrating by parts to obtain

\[
\int_0^{t_n} y dF(y) = - t_n P(X_1 > t_n) + \int_0^{t_n} (1 - F(y)) dy,
\]

we see that the right-hand side of (6) approaches zero as \( n \to +\infty \).
Hence, \( P(S_{nn} > t_n) = o(nP(X_1 > t_n)) \), \( (n \to +\infty) \). But then the lemma and inequality (5) imply that

(7) \( \lim \sup_n (P(S_n > t_n) / \{ nP(X_1 > t_n) \}) \leq 1. \)

**Remark 1.** The proof of the theorem goes through if \( t_n \to +\infty \) independently of \( n \). That is, if \( 1 - F \) is slowly varying then \( P(S_n > t) \sim nP(X_1 > t) \sim P(\max_{i \in S} X_i > t) \) as \( t \to +\infty \) for each \( n \). This known result is given in \([1, p. 272]\).

**Remark 2.** The converse of the theorem is not true; it is easy to show that (1) holds for the one-sided stable law with parameter \( 1/2 \): \( F(x) = 2(1 - \Phi(1/\sqrt{x})) \), \( x > 0 \), where \( \Phi \) is the standard normal d.f., provided that \( n^2 = o(t_n) \) \( (n \to +\infty) \). However, \( 1 - F \) is a regularly varying function with exponent \(-1/2\).

**Remark 3.** If \( 1 - F \) is regularly varying with exponent \(-\gamma\), \( 0 < \gamma < 1 \), then the above argument yields

\[
1 \leq \lim \inf_n \{ P(S_n > t_n) / nP(X_1 > t_n) \} \leq \lim \sup_n \{ P(S_n > t_n) / nP(X_1 > t_n) \} \leq (1 - \gamma)^{-1},
\]

if \( nP(X_1 > t_n) \to 0 \), as \( n \to +\infty \).
References


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