COMMUTATIVITY AND COMMON FIXED POINTS IN RECURRENCE THEORY

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Let \( N \) be the set of nonnegative integers and, for \( e \in N \), let \( \phi_e \) be the partial recursive function of one argument having index \( e \). In 1938 [1, The Recursion Theorem] Kleene showed that if \( f \) is any recursive function then, for some number \( c \), \( \phi_c \cong \phi_f(c) \). It follows that if \( W_e \) (the recursively enumerable (r.e.) set with index \( e \)) is defined as the domain of \( \phi_e \), then \( W_e = W_f(c) \). Call a number-theoretic function \( h \) well-defined on the r.e. sets if, for all \( m, n \in N \), \( W_m = W_n = W_{h(m)} = W_{h(n)} \). In this paper we show that if \( f, g \) are recursive functions which are well-defined on the r.e. sets and which commute as maps of the r.e. sets (i.e., for all \( n \in N \), \( W_{f(g(n))} = W_{g(f(n))} \)), then they have a common fixed point (i.e., for some \( e \in N \), \( W_e = W_{f(e)} = W_{g(e)} \)). We also give an example which shows that the assumption of well-definedness cannot be eliminated.

First we prove a lemma related to the Myhill-Shepherdson Theorem [2, p. 359, Theorem XXIX (6)]. From now on, whenever \( f \) is well defined on the r.e. sets and \( W \) is an r.e. set, we shall write \( f(W) \) for \( W(f) \) where \( e \) is any number such that \( W = W_e \).

**Lemma.** If \( f \) is a partial recursive function well-defined on the r.e. sets and \( W \) is an r.e. set, then

\[
\{f(W)\} = \bigcup \{f(F) \mid F \subseteq W \text{ and } F \text{ is finite}\}.
\]

**Proof.** Our proof consists of two applications of Kleene's Recursion Theorem.

\( f(W) \subseteq \bigcup \{f(F) \mid F \subseteq W\} : \) Suppose \( n \in f(W) \). Let \( h \) be a recursive function such that, for all \( x \),

\[
W_{h(x)} = W \quad \text{if} \quad n \notin f(W_x),
\]

\[
= \text{some finite subset} \quad F \quad \text{of} \quad W \quad \text{if} \quad n \in f(W_x),
\]

and choose \( c \) satisfying \( W_c = W_{h(x)} \). Clearly, \( W_c \subseteq W \). Suppose \( n \in f(W_c) \). Then \( W_c = W \), a contradiction, since \( n \notin f(W) \). So \( n \notin f(W_c) \). But then \( W_c \) is finite, so \( n \in \bigcup \{f(F) \mid F \subseteq W\} \).

\( \bigcup \{f(F) \mid F \subseteq W\} \subseteq f(W) : \) Suppose \( F \subseteq W \) and \( n \in f(F) \). Let \( h \) be a recursive function such that, for all \( x \),

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\[ W_{h(z)} = F \quad \text{if } n \in f(W_z), \]
\[ = W \quad \text{if } n \not\in f(W_z), \]

and choose \( c \) satisfying \( W_c = W_{h(c)} \). Suppose \( n \in f(W_c) \). Then \( W_c = F \), a contradiction, since \( n \in f(F) \). So \( n \not\in f(W_c) \). But then \( W_c = W \), so \( n \in f(W) \).

**Theorem.** Let \( f, g \) be recursive functions such that

\[ W_m = W_n \rightarrow (W_{f(m)} = W_{f(n)} \& W_{g(m)} = W_{g(n)} \& W_{f(g(m))} = W_{f(g(n))}). \]

Let \( V = \bigcup_{k \geq 0} (f \circ g)^k(\emptyset) \). Then \( V \) is r.e., \( V = f(V) = g(V) \) and, for all \( V' \), if \( V' = f(V') = g(V') \), then \( V' \supseteq V \).

**Proof.** Let \( V_0 = \emptyset \), \( V_{n+1} = f(g(V_n)) \). Since \( W_0 = \emptyset \), if we define \( k(0) = 0 \); \( k(n+1) = f(g(k(n))) \), then \( V_n = W_{k(n)} \) for all \( n \geq 0 \). So

\[ V = \bigcup_{n>0} V_n = \bigcup_{n>0} W_{k(n)} \]

is an r.e. set.

If \( h \) is well defined on the r.e. sets and \( W, W' \) are r.e. sets with \( W \subseteq W' \), then, by the lemma,

\[ h(W) = \bigcup h(F)(F \subseteq W) \subseteq \bigcup h(F)(F \subseteq W') = h(W'). \]

We have \( \emptyset \subseteq f(\emptyset) \), \( \emptyset \subseteq g(\emptyset) \). So

\[ g(\emptyset) \subseteq g(f(\emptyset)) = f(g(\emptyset)), \quad f(\emptyset) \subseteq f(g(\emptyset)). \]

Hence \( V_0 \subseteq f(V_0) \subseteq V_1 \), \( V_0 \subseteq g(V_0) \subseteq V_1 \). Inductively, assume \( V_n \subseteq g(V_n) \subseteq V_{n+1} \), \( V_n \subseteq f(V_n) \subseteq V_{n+1} \). Then

\[ V_{n+1} = f(g(V_n)) \subseteq f(V_{n+1}), \]
\[ V_{n+1} = g(f(V_n)) \subseteq g(V_{n+1}), \]

so that

\[ g(V_{n+1}) \subseteq g(f(V_{n+1})) = f(g(V_{n+1})) = V_{n+2} \]

and

\[ f(V_{n+1}) \subseteq f(g(V_{n+1})) = V_{n+2}. \]

Thus

\[ V_{n+1} \subseteq f(V_{n+1}) \subseteq V_{n+2}, \quad V_{n+1} \subseteq g(V_{n+1}) \subseteq V_{n+2}. \]

So, for all \( n \geq 0 \), \( V_n \subseteq f(V_n) \subseteq V_{n+1} \), \( V_n \subseteq g(V_n) \subseteq V_{n+1} \).

This gives

\[ V = \bigcup_{n>0} V_n = \bigcup_{n>0} f(V_n) = \bigcup_{n>0} g(V_n). \]
So
\[ f(V) = \bigcup_{n>0} f(F) (F \subseteq V) = \bigcup_{n>0} \left( \bigcup_{n>0} f(F) (F \subseteq V_n) \right) = \bigcup_{n>0} f(V_n) = V, \]
and
\[ g(V) = \bigcup_{n>0} g(F) (F \subseteq V) = \bigcup_{n>0} \left( \bigcup_{n>0} g(F) (F \subseteq V_n) \right) = \bigcup_{n>0} g(V_n) = V. \]

Also, \( V \) is the least common fixed point. For let \( V' \) be any other common fixed point. Trivially \( V_0 = \emptyset \subseteq V' \); suppose \( V_n \subseteq V' \). Then \( g(V_n) \subseteq g(V') = V' \), so that
\[ V_{n+1} = f(g(V_n)) \subseteq f(V') = V'. \]

Hence \( V = \bigcup_{n>0} V_n \subseteq V' \).

The reader will detect a close connection between the above proof and Kleene’s proof of his “first” recursion theorem [1, p. 66].

There is a version of Theorem 2 for partial recursive functions, rather than r.e. sets. One can replace \( W \) by \( \phi, = \) by \( \sim \), \( \bigcup \) by “least common extension of” and \( \exists \) by “extends.”

We note the assumption of well-definedness in Theorem 2 is necessary. For let \( e_n \) be the Gödel number of the finite system of equations \( h(n) = n \) where \( h \) is a function letter and \( n \) is the numeral for \( n \). Then, for all \( n \), \( W_{e_n} = \{ n \} \). Define
\[ f(x) = \mu y (y \geq x \& (\exists n) (y = e_n)), \]
\[ g(x) = \mu y (y \geq x \& (\exists n) (x \leq e_n \& y = e_{n+1})). \]

Then \( f(g(x)) = g(f(x)) = g(x) \), so \( f \) and \( g \) commute as number-theoretic functions. \( f \) and \( g \) are recursive, but for all \( x \), \( W_{f(x)} \neq W_{g(x)} \). Thus \( f \) and \( g \) cannot possibly have a common fixed point.

References


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