ON THE EXTENSION OF LINEARLY INDEPENDENT SUBSETS OF FREE MODULES TO BASES

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Introduction. In this note we discuss a class of rings with identity with the following property:

(1) Each linearly independent subset of a (unitary) free right \( A \)-module can be extended to a basis, by adjoining elements of a given basis. In view of (1) we call such rings right-Steinitz rings. We prove the equivalence of (1) and the following condition:

(2) Let \( R_1 = \{ \text{x \in A} | \text{x does not have a left inverse} \} \). If an infinite matrix \( T \) of elements of \( R_1 \) is column-finite and if \( T_{ij} = 0 \) for all \( i \leq j \), then, for each \( j \), there is an integer \( N \) such that \( (T + T^2 + \cdots + T^n)_{j+N,j} = 0 \) for all \( n > N \).

To prove the equivalence of (1) and (2) we need to establish several other properties of right-Steinitz rings, which in turn reveal them as being either examples or “near-examples” of classes of rings studied by a variety of investigators, the following cases being representative.

In [1], P. M. Cohn discusses a sequence of three progressively stronger conditions, the strongest being

III. Any generating set with \( n \) elements of a rank \( n \) free module is free. An inductive argument shows that right-Steinitz rings do indeed satisfy the condition. It also follows from the discussion below that right-Steinitz rings satisfy all conditions of Goldie’s local-rings except that the intersection of all powers of the ideal of nonunits may not be zero (cf., e.g., [2]). Obviously, division rings are right-Steinitz rings. If \( Z \) is the ring of integers and if \( p \) is any prime, then \( Z/(p^i) \) satisfies condition (2) as is easily seen. For any field \( \Delta \) and a vector-space \( V \) over \( \Delta \), let \( A = \Delta \times V \), with operations defined by

\[
\begin{align*}
(\delta_1, x_1) + (\delta_2, x_2) &= (\delta_1 + \delta_2, x_1 + x_2) \\
(\delta_1, x_1)(\delta_2, x_2) &= (\delta_1\delta_2, x_1\delta_2 + x_2\delta_1), \quad \delta_i \in \Delta, \quad x_i \in V.
\end{align*}
\]

Then, \( V \) is the ideal of nonunits, with \( V^2 = 0 \), and again condition (2) is easily seen to be satisfied. Another property of right-Steinitz rings is the following: if \( \{ x_i \}_{i=0}^{n} \) is a sequence of nonunits, then, for some index \( n \), \( x_n \cdot x_{n-1} \cdots x_1 = 0 \). Thus, let \( F_0 \) be a division-ring, and let \( F_0[x] \) be the polynomial-ring in one variable over \( F_0 \). Define \( F_i = xF_0[x] / x^{i+1}F_0[x] \) for \( i \geq 1 \). Let \( R \) be the weak direct sum of rings

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Then, \( R \) is a two-sided vector-space over \( F_0 \). Take \( A = F_0 \times R \), with operations
\[
(\delta_1, x_1) + (\delta_2, x_2) = (\delta_1 + \delta_2, x_1 + x_2),
\]
\[
(\delta_1, x_1) \cdot (\delta_2, x_2) = (\delta_1 \delta_2, x_1 \delta_2 + \delta_1 x_2 + x_1 x_2), \quad \delta_i \in F_0, \quad x_i \in R.
\]
Notice that if \( x_0 = (\alpha, \cdots, \alpha_{i+1}, 0, \cdots, 0, \cdots) \in R \), then given any sequence \( \{x_i\}_{i=0}^{\infty} \) of elements in \( R \), \( x_i x_{i-1} \cdots x_0 = 0 \). From this, again, condition (2) follows. Notice that in this case the ideal \( R \) is not nilpotent, while if \( R \) is nilpotent, (2) follows easily.

Clearly, if \( T \) is an infinite proper triangular matrix, i.e., a triangular matrix with 0 diagonal, over any ring, then the inverse of \( I - T \) exists and is equal to \( I + T + T^2 + \cdots \). The argument depends on the fact that \( I - T \) as well as \( I + T + T^2 + \cdots \) are row finite and because \((T^n)_{ij} = 0 \) if \( n > i - j \). We can thus restate condition (2) to obtain the equivalent form:

(2)' If \( T \) is an infinite column-finite proper triangular matrix of elements of \( R_1 \), so is \((I - T)^{-1}\). In concluding this introduction we should like to thank the referee for several helpful comments and a simplification of the proof of Theorem 2.

The equivalence of conditions (1) and (2). Note that all modules under discussion are right unitary.

**Lemma 1.** If \( A \) satisfies (1), then for each infinite sequence \( \{x_i\}_{i=0}^{\infty} \) of elements of \( A \) which do not have a left inverse, there is a nonnegative integer in such that \( x_n x_{n-1} \cdots x_0 = 0 \).

**Proof.** Let \( \{u_i\}_{i=0}^{\infty} = U \), be a basis for a free \( A \)-module \( M \), i.e \( M = [U] = [U_i]_{i=0}^{\infty}. \) Let \( v_i = u_i - u_{i+1} x_i, \quad i = 0, 1, 2, \cdots \). Then, \( \{v_i\}_{i=0}^{\infty} \) is linearly independent. Indeed, \( \sum_{i=0}^{\infty} a_i v_i = 0 \) implies \( \sum_{i=0}^{\infty} (u_i - u_{i+1} x_i) a_i = 0 \), i.e.,
\[
u_0 a_0 + u_1 (a_1 - x_0 a_0) + \cdots + u_s (a_s - x_{s-1} a_{s-1}) - u_{s+1} x_s a_s = 0,
\]
whence \( a_0 = a_1 = \cdots = a_s = 0 \). Now let \( V \) be the submodule spanned by \( \{v_i\}_{i=0}^{\infty} \). Since \( \{v_i\}_{i=0}^{\infty} \) can be extended to a basis of \( M \) by adjoining elements of \( U \), suppose \( \{v_i\}_{i=0}^{\infty} \cup \{u_i, u_{i+1}, \cdots \} \) is a basis of \( M \).

Then \( u_i \equiv u_i y \) (mod \( V \)) for some \( y \in A \) if \( i_1 < i_2 \), whence \( u_i \in \operatorname{span}(v, \{u_i\}) \). Thus \( \{v_i\}_{i=1}^{\infty} \cup \{u_i\} \) must be a basis for some \( u_i \in U \). Then, \( u_{i+1} \equiv u_{i+1} x_t \) (mod \( V \)), \( u_i \equiv u_{i+1} x_t \) (mod \( V \)). Hence, \( u_i \equiv u_{i+1} x_t \) (mod \( V \)), i.e., \( 1 - a x_t = 0 \). Since \( x_t \) does not have a left inverse, \( V = M \), and \( \{v_i\}_{i=0}^{\infty} \) is a basis of \( M \). Thus, if \( \sum_{i=0}^{\infty} v_i b_i = u_0 \), i.e.,
\[
u_0 b_0 + u_1 (b_1 - x_0 b_0) + \cdots + u_s (b_s - x_{s-1} b_{s-1}) - u_{s+1} x_s b_s = u_0,
\]
we have \( b_0 = 1, b_1 = x_0, \ldots, b_s = x_{s-1} \cdots x_0, x_s b_s = x_s \cdots x_0 = 0. \) Hence, \( n = n(s) = s \) and the lemma follows.

**Lemma 2.** Let \( A \) be a ring with identity, \( R_1 = \{ x \in A \mid x \text{ does not have a left inverse} \} \) and \( R_2 = \{ x \in A \mid x \text{ does not have a right inverse} \} \). If every element of \( R_1 \) is nilpotent then \( R_1 = R_2 \) and \( R_1 \) forms the unique maximal ideal of \( A \).

**Proof.** First we show that \( R_1^c = A \setminus R_1 \) forms a group. It is clear that \( R_1^c \) is closed under multiplication. Suppose \( x \in R_1^c \), then there is a \( y \in A \) such that \( y \cdot x = 1 \). If \( y \in R_1^c \) then there is an integer \( n \) such that \( y^n = 0, y^{n-1} \neq 0 \). Hence, \( 0 = y^n \cdot x = y^{n-1} \), and this is a contradiction. So, \( y \in R_1^c \). Therefore, if \( yx = 1 \) then \( xy = 1 \). Thus, \( R_1^c \subseteq R_2^c \). Hence \( R_1 \supseteq R_2 \). Suppose \( x \in R_1 \) and \( x \in R_2 \), then there is a \( y \in A \) such that \( xy = 1 \). Since \( x \) is nilpotent, this is also a contradiction. Hence \( R_1 = R_2 \). To show \( R_1 \) is closed under +, let \( x \) and \( y \) be elements of \( R_1 \), and suppose \( x + y \notin R_1 \). Then there is a \( z \in A \) such that \( z(x + y) = 1 \), \( zx + zy = 1 \), \( zx = 1 - zy \). Since \( zy \in R_1 \), it is nilpotent and \( 1 - zy \) has an inverse, i.e., \( zx \) has an inverse. This is a contradiction. Hence \( R_1 \) is closed under +. It is clear that \( zR_1 \subseteq R_1 \) for any \( z \in A \). Also if \( x \in R_1, z \in A \) and \( zx \in R_1 \), then there is a \( y \in A \) such that \( yxz = 1 \). This is a contradiction because \( yx \) is nilpotent. Hence \( R_1 \) is an ideal of \( A \). It is clear that \( R_1 \) is the unique maximal ideal of \( A \) because \( R_1^c \) consists of the units of \( A \). In short, since \( R_1 \) forms a left ideal, Lemma 2 follows as is well known.

**Corollary 3.** If, for each infinite sequence \( \{x_i\}_{i=0}^\infty \) of elements of \( A \) which do not have a left inverse there is a nonnegative integer \( n \) such that \( x_n \cdot x_{n-1} \cdots x_0 = 0 \), then there is a nonzero element \( a \) of \( A \) such that \( b \cdot a = 0 \) for all elements \( b \) of \( A \) which do not have a right inverse.

**Proof.** Let \( R_2 \) be the collection of all nonzero elements of \( A \) which do not have a right inverse. If, for all nonzero elements \( x \) of \( R_2 \), \( R_3 x \neq \{ 0 \} \), we have a choice function \( f: R_2 \backslash 0 \rightarrow R_2 \backslash 0 \), such that \( (x)f \cdot x \neq 0 \), whence each nonzero \( x_1 \) in \( R_2 \) generates an infinite sequence \( \{x_1, \ldots, x_n, \ldots\} \) with \( x_1 = (x_{i-1} \cdots x_1)f \), such that \( x_n \cdot x_{n-1} \cdots x_1 \neq 0 \) for each integer \( n \). This is a contradiction. Hence, since \( R_2 = \{ 0 \} \) implies \( R_2 \cdot 1 = 0 \), the lemma follows.

**Theorem 1.** If a ring \( A \) satisfies (1), then it satisfies (2).

**Proof.** From Lemma 1 and Lemma 2, \( A \) has a unique maximal ideal consisting of all nonunits, \( R_1 = R_2 = R \). Let \( T \) be a matrix provided by (2), and let
let $u = \{u_j \mid j = 1, 2, \cdots \}$ be a basis of a free $A$-module $M$. Let

$$v_j = u_j - \sum_i u_i T_{ij} \quad \text{for } j = 1, 2, 3, \cdots,$$

then clearly $V = \{v_j \mid j = 1, 2, 3, \cdots \}$ is a linearly independent subset of $M$. From the corollary to Lemma 2, there is a nonzero element $a$ such that $Ra = 0$. Hence $v_i a = U_j a$ for each $j$. Since we suppose that $A$ satisfies (1) and $V$ is a basis of $M$. Suppose $\sum_{j=1}^n v_j S_{ji} = u_i$ and $S_{ji} \subseteq A$ for each $i$. Let $s$ be the matrix whose elements are $S_{ji}$, then $(I-T)s = 1$ where $I_{ij} = \delta_{ij}$, where as mentioned before, $S = I + T + T^2 + \cdots$.

**Lemma 3.** Let $A$ be a ring satisfying condition (2), then for each sequence $\{x_i\}_{i=1}^n$ of elements of $R_1$, there is an $n$ such that

$$x_n \cdot x_{n-1} \cdots x_1 = 0.$$

**Proof.** Consider the case $T_{ij} = x_j$ if $i = j+1$ and $T_{ij} = 0$ if $i \neq j+1$. Then

$$(T^n)_{n,1} = T_{n+1,n} \cdot T_{n,n-1} \cdots T_{2,1} = x_n \cdot x_{n-1} \cdots x_1.$$ Hence from condition (2), $x_n \cdot x_{n-1} \cdots x_1 = 0$.

**Lemma 4.** If $A$ satisfies condition (2) then $R_1 = R_2 = R$ and if $R \neq \{0\}$ there is a nonzero element $a \in R$ such that $Ra = 0$, and $R$ is the unique maximal ideal from Lemmas 3, 2 and the corollary to Lemma 2.

**Lemma 5.** Let $A$ be a ring as in the corollary to Lemma 2, then any finite linearly independent subset of a free $A$-module $M$ can be extended to a basis by adjoining elements of a given basis.

**Proof.** Let $V = \{v_1, v_2, \cdots, v_n\}$ be a linearly independent set, and $U = \{u_i \mid i \in \Lambda\}$ be a basis of $M$. Let $v_1 = \sum u_i a_i$ for $a_i \in A$, then not all $a_i$ are elements of $R$, otherwise $v_1 a = \sum u_i a_i a = 0$, where $a$ is the element of $A$ of Corollary 3. Let $a_1 \in R$, then $v_1 = (v_1 - \sum_{i>1} u_i a_i) a_1^{-1}$, hence $\{v_1\} \cup \{u_i \mid i \neq 1\}$ is a basis. Suppose $\{v_1, v_2, \cdots, v_{n-1}\} \cup \{u_i \mid i > n\}$ is a basis and $v_n = \sum_{i<n} v_i b_i + \sum_{i>n} u_i a_i$, then not all $a_i$ are in $R$, otherwise $v_n a = \sum_{i<n} v_i b_i a$. Hence $v_1, v_2, \cdots, v_n$ can be extended to a basis by adjoining some elements of $U$. Therefore, by induction, the lemma is proved.

**Theorem 2.** If a ring $A$ satisfies (2) then it satisfies (1).

**Proof.** Let $U = \{u_i \mid i \in \Lambda\}$ be a basis of $M$, and $V = \{v_j \mid j \in T\}$ be a linearly independent subset. Without loss of generality we may
assume that $V$ is a maximal linearly independent subset of $V \cup U$. Suppose $[V] \neq M$, then there is a $u_1 \in [V]$. Let $u_1 c = \sum_{j=1}^{n} V_j b_j$, for some $c \in A$ and $b_j \in A$. Since $\{v_1, v_2, \ldots, v_n\}$ can be extended to a basis by adjoining some elements of $U$,

$$u_1 = \sum_{j} v_j b_j + \sum_{i} u_i T_{ii} \quad \text{for some } b_j, \quad T_{ii} \in A,$$

whence $b_j c = b_j$ and $T_{ii} c = 0$ for all $j$ and $l$. Hence $T_{ii} \in R$ and $u_1 = \sum_{l=1}^{2} u_i T_{ii} \mod [V]$. If $T_{ii} \neq 0$, then $u_1 \neq \sum_{l=2}^{2} u_i T_{ii} (1 - T_{ii})^{-1} \mod [V]$, and we may thus assume $u_1 = \sum_{l=2}^{2} u_i T_{ii} \mod [V]$. Repeating this argument, we obtain a countably infinite column-finite matrix $T$ of elements of $R$ such that $T_{ii} = 0$ if $l \leq i$ and $u_i = \sum_{l=2}^{2} u_i T_{ii} \mod [V]$. By $(2)'$, $S = (I - T)^{-1}$ is column-finite. If $X$ denotes the row matrix $(u_1, u_2, \ldots)$, then $X(I - T) \equiv 0 \mod [V]$ implies $(X(I - T)) S \equiv 0 \mod [V]$, contradicting the fact that $u_1 \in [V]$.

**Corollary.** If a ring $A$ satisfies $(2)$, then, for any $A$-module $M$, $M = MR$ implies $M = \{0\}$.

**Proof.** Let $\{u_i | i \in T\}$ be a generating set, then for each $u_i$, $u_i = \sum_{i} u_i T_{ii}$ where $T_{ii} \in R_i$. We can assume that $T = \{1, 2, \ldots\}$ and $T_{ii} = 0$ if $l \geq i$ as before. Then, $u_i - \sum_{i} u_i T_{ii} = 0$ implies $u_i = 0$ for each $i$.

**References**


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