APPLICATIONS OF AN INEQUALITY FOR THE SCHUR COMPLEMENT

EMILIE V. HAYNSWORTH

1. Introduction. Suppose $B$ is a nonsingular principal submatrix of an $n \times n$ matrix $A$. The Schur Complement of $B$ in $A$, denoted by $(A/B)$, is defined as follows: Let $\tilde{A}$ be the matrix obtained from $A$ by the simultaneous permutation of rows and columns which puts $B$ into the upper left corner of $\tilde{A}$,

$$A = \begin{pmatrix} B & C \\ D & G \end{pmatrix},$$

leaving the rows and columns of $B$ and $G$ in the same increasing order as in $A$. Then the Schur Complement of $B$ in $A$ is

$$(A/B) = G - DB^{-1}C.$$  

Schur proved that the determinant of $A$ is the product of the determinant of any nonsingular principal submatrix $B$ with that of its Schur complement,

$$(A/B) = \det(B) \det(A/B).$$  

The inertia of an Hermitian matrix $A$ is given by the ordered triplet, $\text{In} A = (\pi, \nu, \delta)$, where $\pi$ denotes the number of positive, $\nu$ the number of negative, and $\delta$ the number of zero roots of the matrix $A$. In a previous paper [2], it was shown that the inertia of an Hermitian matrix can be determined from that of any nonsingular principal submatrix together with that of its Schur complement. That is, if $A$ is Hermitian, and $B$ is a nonsingular principal submatrix of $A$, then

$$\text{In} A = \text{In} B + \text{In}(A/B).$$

More recently, the author, with Douglas Crabtree [1], proved the identity,

$$(A/B) = ((A/C)/(B/C)).$$

In Theorem 1 of §2 we make use of (3) to prove an extension of a theorem by Marcus [3]. Then in Theorem 2 we apply the result of Theorem 1 to obtain an inequality for the Schur complement which is similar to Minkowski's famous inequality (see [4]) for the determinant of the sum of positive definite Hermitian matrices:

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\[ |A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n} \] (Minkowski).

This, of course, implies

(4) \[ |A + B| \geq |A| + |B|. \]

A number of extensions of the Minkowski inequality have been proved by Marcus, Minc and others (see [5]).

In Theorem 3 we obtain some new inequalities for the determinant of the sum of two positive definite Hermitian matrices.

2. An extension of a theorem by Marcus. In a recent paper [3] M. Marcus proved a number of interesting inequalities for positive definite Hermitian matrices, including the following: If \( H \) and \( K \) are positive definite matrices of order \( n \), and \( X \) and \( Y \) are arbitrary vectors, then

\[(H^{-1}X, X) + (K^{-1}Y, Y) \geq ((H + K)^{-1}(X + Y), (X + Y)).\]

It is shown in Theorem 1 that by making use of the properties of the Schur complement this inequality can be extended to the case where \( X \) and \( Y \) are arbitrary \( n \times m \) matrices. We shall use the notation \( A \geq 0 \) for a positive semidefinite matrix (p.s.d. matrix), with strict inequality implying that \( A \) is positive definite (p.d.). If \( A \) and \( B \) are p.s.d. matrices, the statement \( A \geq B \) will mean \( A - B \geq 0 \).

**Theorem 1.** Suppose \( H \) and \( K \) are positive definite matrices of order \( n \). Then if \( X \) and \( Y \) are arbitrary \( n \times m \) matrices, the \( m \times m \) matrix

(5) \[ Q = X^*H^{-1}X + Y^*K^{-1}Y - (X + Y)^*(H + K)^{-1}(X + Y) \]

is positive semidefinite.

**Proof.** Let \( A \) and \( B \) be the Hermitian matrices of order \( 2n \),

\[ A = \begin{pmatrix} H & X \\ X^* & X^*H^{-1}X \end{pmatrix}, \quad B = \begin{pmatrix} K & Y \\ Y^* & Y^*K^{-1}Y \end{pmatrix}. \]

From (3), it is clear that a nonzero Hermitian matrix is positive semidefinite (definite) if and only if there exists a positive definite principal submatrix whose Schur complement is positive semidefinite (definite). Thus, by inspection, the matrices \( A \) and \( B \) are positive semidefinite. Then, since the sum of any two positive semidefinite matrices is also positive semidefinite (or definite) we have

\[ A + B = \begin{pmatrix} H + K & X + Y \\ X^* + Y^* & X^*H^{-1}X + Y^*K^{-1}Y \end{pmatrix} \geq 0. \]
This proves the theorem, as the matrix $Q$ in (5) is the Schur complement of $H+K$ in $A+B$.

3. An inequality for the Schur complement.

**Theorem 2.** Suppose $A$ and $B$ are Hermitian matrices of order $n$, partitioned into $2\times2$ block matrices, $A = (A_{ij})$, $B = (B_{ij})$, $i, j = 1, 2$, where $A_{11}$ and $B_{11}$ are square of order $m$. If $A \succ 0$, $B \succeq 0$, $A_{11} > 0$, $B_{11} > 0$, then

$$
(A + B/A_{11} + B_{11}) \succeq (A/A_{11}) + (B/B_{11}).
$$

**Proof.** By the previous arguments, $A_{11} + B_{11} > 0$, and $A + B \succeq 0$. From the definition,

$$
(A + B/A_{11} + B_{11}) = (A_{22} + B_{22}) - (A_{21} + B_{21})(A_{11} + B_{11})^{-1} \cdot (A_{12} + B_{12}).
$$

By Theorem 1,

$$
(A_{21} + B_{21})(A_{11} + B_{11})^{-1}(A_{12} + B_{12}) \leq A_{21}A_{11}^{-1}A_{12} + B_{21}B_{11}^{-1}B_{12}.
$$

Thus

$$
(A + B/A_{11} + B_{11}) \succeq (A_{22} + B_{22}) - (A_{21}A_{11}^{-1}A_{12} + B_{21}B_{11}^{-1}B_{12}) = (A/A_{11}) + (B/B_{11}).
$$

This proves the formula (6), which we now apply to find a new inequality for the determinant of the sum of two positive definite Hermitian matrices.

4. Some determinantal inequalities.

**Theorem 3.** Suppose $A$ and $B$ are positive definite Hermitian matrices. Let $A_k$ and $B_k$, $k = 1, \cdots, n$, denote the principal submatrices of order $k$ in the upper left corner of the matrices $A$ and $B$ respectively. Then

$$
| A + B | \geq | A | \left( 1 + \sum_{k=1}^{n-1} \frac{| B_k |}{| A_k |} \right) + | B | \left( 1 + \sum_{k=1}^{n-1} \frac{| A_k |}{| B_k |} \right).
$$

**Corollary.** If $A$ and $B$ are positive definite, and $A \succ B$, then

$$
| A + B | > | A | + n | B |.
$$

For the proof of Theorem 3 we need the following lemmas. Lemma 1 is probably well known, as it follows immediately from the Minkowski inequality (4). Lemma 2 follows as a corollary to Lemma 1 and Theorem 2.
Lemma 1. If $A$ and $B$ are positive definite Hermitian matrices and $A \succ B$, then $|A_k| > |B_k|$, $k = 1, \cdots, n$.

Proof. Let $A - B = C \succ 0$. Then $A_k = B_k + C_k$ ($k = 1, \cdots, n$) where $A_k$, $B_k$, and $C_k$ are positive definite, since they are principal submatrices of positive definite matrices. Then by (4), $|A_k| \geq |B_k| + |C_k| > |B_k|$ ($k = 1, \cdots, n$).

Lemma 2. If $A$ and $B$ satisfy the conditions of Theorem 2, then
\[ |(A + B/A_{11} + B_{11})| \geq |A|/|A_{11}| + |B|/|B_{11}|. \]

Proof. By Theorem 2 and Lemma 1,
\[ |(A + B/A_{11} + B_{11})| \geq |(A/A_{11}) + (B/B_{11})| \]
\[ \geq |(A/A_{11})| + |(B/B_{11})| \quad \text{by (4)} \]
\[ = |A|/|A_{11}| + |B|/|B_{11}| \quad \text{by (2)}. \]

Proof of Theorem 3. We prove the theorem by induction on $n$. For $n = 2$, we have from (2),
\[ (9) \quad |A + B| = |A_1 + B_1| \quad |(A + B/A_1 + B_1)|. \]

By Lemma 2,
\[ |(A + B/A_{11} + B_{11})| \geq |A|/|A_{11}| + |B|/|B_{11}|. \]

Thus, using (4) on the first factor on the right in (9),
\[ |A + B| \geq (|A_1| + |B_1|)(|A|/|A_{11}| + |B|/|B_{11}|) \]
which proves (7) for $n = 2$.

Now assume (7) holds for matrices of order less than or equal to $n - 1$. Then, if $A$ and $B$ are of order $n$,
\[ |A + B| \geq (|A_{n-1} + B_{n-1}|) \quad |(A + B/A_{n-1} + B_{n-1})|, \]
where, by the inductive assumption,
\[ |A_{n-1} + B_{n-1}| \]
\[ \geq |A_{n-1}| \left(1 + \sum_{k=1}^{n-2} \frac{|B_k|}{|A_k|}\right) + |B_{n-1}| \left(1 + \sum_{k=1}^{n-2} \frac{|A_k|}{|B_k|}\right), \]
and, by Lemma 2,
\[ |(A + B/A_{n-1} + B_{n-1})| \geq |A|/|A_{n-1}| + |B|/|B_{n-1}|. \]

Thus
\[ |A + B| \geq \left( |A_{n-1}| \left( 1 + \sum_{k=1}^{n-2} \frac{|B_k|}{|A_k|} \right) \right. \]
\[ + |B_{n-1}| \left( 1 + \sum_{k=1}^{n-2} \frac{|A_k|}{|B_k|} \right) \left( \frac{|A|}{|A_{n-1}|} + \frac{|B|}{|B_{n-1}|} \right) \]
\[ = |A| \left( 1 + \sum_{k=1}^{n-2} \frac{|B_k|}{|A_k|} \right) + |B| \left( 1 + \sum_{k=1}^{n-2} \frac{|A_k|}{|B_k|} \right) \]
\[ + \frac{|A_{n-1}|}{|B_{n-1}|} \left( 1 + \sum_{k=1}^{n-2} \frac{|B_k|}{|A_k|} \right) |B| \]
\[ + \frac{|B_{n-1}|}{|A_{n-1}|} \left( 1 + \sum_{k=1}^{n-2} \frac{|A_k|}{|B_k|} \right) |A| \]
\[ \geq |A| \left( 1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|} \right) + |B| \left( 1 + \sum_{k=1}^{n-1} \frac{|A_k|}{|B_k|} \right). \]

This proves Theorem 3.

The corollary follows as an immediate consequence of Lemma 1, since if \(A > B\),
\[ \frac{|A_k|}{|B_k|} > 1 \quad (k = 1, \ldots, n). \]
Hence
\[ |A + B| \geq |A| \left( 1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|} \right) + n |B| \geq |A| + n |B|. \]

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**References**