

OPERATIONS PRESERVING ALL EQUIVALENCE RELATIONS

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In the recent book by R. S. Pierce [1, p. 38] the following theorem is given as an exercise. If f is a finitary operation on a set A of cardinal > 2 and f has the substitution property relative to every equivalence relation on A , then either f is a constant or f is a projection (that is, there exists an i such that $f(x) = x(i)$ for all x in the domain of f). We will determine all operations f , finitary or not, with this substitution property.

DEFINITION. $|J|$ denotes the cardinal of J . If β is a cardinal, β^+ is the next larger cardinal.

DEFINITION. If R is an equivalence relation on A and $x \in A^\alpha$, $y \in A^\alpha$, then we write xRy whenever $(x(i), y(i)) \in R$ for all $i \in \alpha$. An α -ary operation f on A is said to have the substitution property relative to R if $(f(x), f(y)) \in R$ whenever xRy .

DEFINITION. If $a \in A$, then R_a denotes the equivalence relation on A whose equivalence classes are $\{a\}$ and $A - \{a\}$.

DEFINITION. If γ is a cardinal, then a filter F of subsets of S is called γ -complete if $\bigcap_{j \in J} C_j \in F$ whenever $|J| < \gamma$ and $C_j \in F$ for all $j \in J$. Every filter is ω -complete.

LEMMA. Let γ be a cardinal > 3 and suppose F is a family of subsets of S . Then F is a γ -complete prime filter if and only if the following condition holds:

(*) Whenever $S = \bigcup_{j \in J} B_j$, $|J| < \gamma$, and the B_j are pairwise disjoint, then $B_j \in F$ for exactly one $j \in J$.

PROOF. If F is a γ -complete prime filter, then the condition is easily seen to hold. Suppose (*) holds. For any $C \in S$, exactly one of C and $S - C$ is in F . Suppose $C \in F$ and $C \subseteq D \subseteq S$. Since $S = C \cup (D - C) \cup (S - D)$, we have $S - D \notin F$. Therefore $D \in F$, and we have shown F is ancestral. Now suppose $C_j \in F$ for all $j \in J$ and J is a well ordered set of cardinal $< \gamma$. If γ is finite we may assume $|J| = 2$ and so in all

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cases, $|J| + 1 < \gamma$. Let $D_j = S - C_j$ and $D = \bigcup_{j \in J} D_j$. Let $B_j = D_j - \bigcup_{i < j} D_i$. Then $B_j \notin F$ for all $j \in J$, $D = \bigcup_{j \in J} B_j$, and the B_j are pairwise disjoint. Since $S = (S - D) \cup \bigcup_{j \in J} B_j$, it follows from (*) that $S - D \in F$. But $S - D = \bigcap_{j \in J} C_j$, and so the proof is complete.

THEOREM. *Let A be a set of cardinal $\beta > 2$, and let f be a nonconstant α -ary operation on A , where α is an ordinal. Then the following are equivalent.*

- (i) *f has the substitution property relative to every equivalence relation on A ;*
- (ii) *f has the substitution property relative to R_a , for all $a \in A$;*
- (iii) *there exists a β^+ -complete prime filter F of subsets of α such that for all $x \in A^\alpha$, $f(x) = a$ such that $\{i : x(i) = a\} \in F$.*

REMARK. If α is finite, or if β is infinite and α is less than the first measurable cardinal, the filter F must be principal and so (iii) is equivalent to the assertion that f is a projection.

PROOF. It is obvious that (i) implies (ii). The proof that (ii) implies (iii) is divided into parts. Assume f satisfies the hypothesis of (ii). We may assume that A is identified with β . We first show

(1) For all $x \in A^\alpha$, $f(x) = x(i)$ for some i .

Suppose for some y , $f(y) = a$ and $a \neq y(i)$ for all i . We show f must be constant. Suppose $f(x) = b \neq a$ for some x . There exists $c \in A$ such that $c \neq a$ and $c \neq b$. Let $z \in A^\alpha$ be defined by

$$\begin{aligned} z(i) &= x(i) && \text{if } x(i) \neq a, \\ &= c && \text{otherwise.} \end{aligned}$$

Then $zR_b x$ and therefore $f(z) = b$. But $zR_a y$, since $z(i) \neq a$ for all i . Hence $(f(z), f(y)) = (b, a) \in R_a$, which is a contradiction.

If B is a subset of α , we define the sequence $x_B \in A^\alpha$ by

$$\begin{aligned} x_B(i) &= 0 && \text{for } i \in B, \\ &= 1 && \text{otherwise.} \end{aligned}$$

Let F be the set of all subsets B of α such that $f(x_B) = 0$. We show

(2) For any $a \in A$ and any $x \in A^\alpha$, we have $f(x) = a$ if and only if $\{i : x(i) = a\} \in F$.

Let $B = \{i : x(i) = a\}$. There exists $b \in A$ such that $b \neq a$ and $b \neq 0$. Let $y(i) = a$ for $i \in B$, and $y(i) = b$ otherwise. Then $xR_a y$, and so $f(x) = a$ if and only if $f(y) = a$. Let $z(i) = 0$ for $i \in B$, and $z(i) = b$ otherwise. Then $yR_b z$, and therefore $f(y) \neq b$ if and only if $f(z) \neq b$. By (1), it follows that $f(y) = a$ if and only if $f(z) = 0$. But $xR_0 x_B$, and so $f(z) = 0$ if and only if $f(x_B) = 0$. Hence $f(x) = a$ if and only if $B \in F$.

Now we are ready to prove (iii). By (1) and (2), it is clear that $\emptyset \notin F$. Suppose the family $\langle B_j : j \in J \rangle$ satisfies the hypothesis of (*) in the Lemma, where $S = \alpha$ and $\gamma = \beta^+$. By adding enough copies of the empty set we may assume $J = \beta$. Let $x \in A^\alpha$ be defined by $x(i) = j$ when $i \in B_j$. Then by (2), $B_j \in F$ if and only if $j = f(x)$. Therefore the condition (*) of the Lemma holds and F is a β^+ -complete prime filter. By (2) we see that (iii) holds.

To show that (iii) implies (i), let R be any equivalence relation on A . Suppose $x \in A^\alpha$, $y \in A^\alpha$ and xRy . Then since $\{i : x(i) = f(x)\}$ and $\{i : y(i) = f(y)\}$ are in F , there exists $i \in \alpha$ such that $x(i) = f(x)$ and $y(i) = f(y)$. Therefore $(f(x), f(y)) = (x(i), y(i)) \in R$, and the proof is complete.

REFERENCE

1. R. S. Pierce, *Introduction to the theory of abstract algebras*, Holt, Rinehart and Winston, New York, 1968. MR 37 #2655.

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