HOMEOMORPHIC MEASURES IN METRIC SPACES

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Abstract. For any nonatomic, normalized Borel measure \( \mu \) in a complete separable metric space \( X \) there exists a homeomorphism \( h: \mathcal{N} \rightarrow X \) such that \( \mu = \lambda h^{-1} \) on the domain of \( \mu \), where \( \mathcal{N} \) is the set of irrational numbers in \( (0, 1) \) and \( \lambda \) denotes Lebesgue-Borel measure in \( \mathcal{N} \). A Borel measure in \( \mathcal{N} \) is topologically equivalent to \( \lambda \) if and only if it is nonatomic, normalized, and positive for relatively open subsets.

1. Definitions and results. A topological measure space is a pair \((X, \mu)\), where \( X \) is a topological space and \( \mu \) is a measure on the class of Borel subsets of \( X \). \((X, \mu)\) is homeomorphic to \((Y, \nu)\) if there exists a homeomorphism of \( X \) onto \( Y \) that makes \( \nu \) correspond to \( \mu \), and then \( \nu \) is said to be topologically equivalent to \( \mu \). If \( B \) is a Borel subset of \((X, \mu)\), then \( \mu_B \) denotes the restriction of \( \mu \) to the class of Borel subsets of \( B \). A measure \( \mu \) is everywhere positive if \( \mu(G) > 0 \) for every nonempty open set \( G \), nonatomic if \( \mu(\{x\}) = 0 \) for each \( x \in X \), and normalized if \( \mu(X) = 1 \).

Let \( \mathcal{N} \) denote the set of irrational numbers in \( I = [0, 1] \), and let \( \lambda \) denote the restriction of Lebesgue measure \( m \) to the Borel subsets of \( \mathcal{N} \). It is known \([8, \text{Theorem 2, p. 886}]\) that a Borel measure in the \( n \)-dimensional cube \( I^n \) is topologically equivalent to \( n \)-dimensional Lebesgue-Borel measure in \( I^n \) if and only if it is everywhere positive, nonatomic, normalized, and vanishes on the boundary. A similar theorem will be shown to hold in \( \mathcal{N} \).

Theorem 1. A topological measure space \((X, \mu)\) is homeomorphic to \((\mathcal{N}, \lambda)\) if and only if \( X \) is homeomorphic to \( \mathcal{N} \) and \( \mu \) is an everywhere positive, nonatomic, normalized Borel measure in \( X \). In particular, any such measure in \( \mathcal{N} \) is topologically equivalent to \( \lambda \).

It is known \([2, \S 6, \text{Exercise 8c, p. 84}]\) that if \( X \) is a compact metric space, and \( \mu \) is a nonatomic, normalized Borel measure in \( X \), then \((X, \mu)\) is almost homeomorphic to \((I, \lambda)\), in the sense that there exist sets \( A \subset I \) and \( B \subset X \) such that \( \lambda(I - A) = 0 \), \( \mu(X - B) = 0 \), and \((B, \mu_B)\) is homeomorphic to \((A, \lambda_A)\). We shall show that this conclusion still holds when \( X \) is a complete separable metric space, and that the set \( A \) can always be taken equal to \( \mathcal{N} \).

Received by the editors July 3, 1969.

AMS Subject Classifications. Primary 2813, 2810, 2870; Secondary 5435, 5460.

Key Words and Phrases. Topologically equivalent Borel measures, homeomorphic measure spaces, measure-preserving mapping, complete separable metric space, space of irrational numbers, Cantor set.

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Theorem 2. If $X$ is a topologically complete separable metric space, and $\mu$ is a nonatomic, normalized Borel measure in $X$, then there exists a $G_\delta$ set $B$ in $X$ such that $\mu(X - B) = 0$ and $(B, \mu_B)$ is homeomorphic to $(\mathfrak{H}, \lambda)$.

Any uncountable complete separable metric space $X$ contains a copy of $\mathfrak{H}$ [6, Corollary 2, p. 352]. Theorem 2 implies that the most general nonatomic, normalized Borel measure in such a space can be constructed by mapping $\mathfrak{H}$ into $X$ by a homeomorphism $h$, and then defining $\mu(E) = \lambda h^{-1}(E)$ for every Borel set $E$. The completion of $\mu$ is equal to $\mu h^{-1}$.

Theorem 2 can be generalized immediately to nonseparable spaces whose separability character has measure zero; such a space has an open set of measure zero whose complement is separable [7, Theorem III, p. 137]. On the other hand, the indispensability of completeness and metrizability is indicated by the following remarks.

Remark 1. A separable metric space with a nonatomic, normalized Borel measure need not contain a copy of $\mathfrak{H}$.

Let $X$ be a subset of $I$ such that both $X$ and $I - X$ meet every nonempty perfect subset of $I$. Then $X$ has outer Lebesgue measure one and inner measure zero. Any relatively Borel subset $A$ of $X$ is of the form $A = X \cap B$, for some Borel set $B$ in $I$. The formula $\mu(A) = m(B)$ defines unambiguously a nonatomic, normalized Borel measure $\mu$ in the separable metric space $X$, but $X$ contains no copy of $\mathfrak{H}$.

Remark 2. A compact Hausdorff space with a nonatomic, normalized, regular Borel measure need not contain a copy of $\mathfrak{H}$, or even of the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \}$.

The Stone space $X$ corresponding to any finite nonatomic measure algebra admits a nonatomic, normalized, regular Borel measure [5, §24]. Since $X$ is compact and basically disconnected, every infinite closed subset contains a copy of $\beta N$ [4, Problem 9H.2, p. 137], and so its cardinal is at least $2^\omega$. The product of uncountably many copies of $(I, m)$ is another example in which every compact metrizable subspace has measure zero [2, §8, Exercise 14a, p. 110].

2. Proofs of Theorems 1 and 2. A metrizable space is homeomorphic to $\mathfrak{H}$ if and only if it is topologically complete, separable, 0-dimensional, and nowhere locally compact [1, Satz IV, p. 95]. Hence any nonempty open subset of $\mathfrak{H}$ is homeomorphic to $\mathfrak{H}$. Likewise, any $G_\delta$ set that is both dense and frontier in some topologically complete, separable, 0-dimensional space $Y$ is homeomorphic to $\mathfrak{H}$ [6, Theorem 3, p. 349].
Lemma 1. If \( \mu \) is an everywhere positive, nonatomic, finite Borel measure in \( \mathfrak{M} \), and if \( \{\alpha_i\} \) is a sequence of positive real numbers such that \( \sum \alpha_i = \mu(\mathfrak{M}) \), then there exists a partition of \( \mathfrak{M} \) into open sets \( U_i \) such that \( \mu(U_i) = \alpha_i \) for all \( i \in \mathbb{N} \).

Proof. Let \( a(i, j) = j\alpha_i/(j+1) \) \((i \in \mathbb{N}, j \in \mathbb{N})\). Let \( \{r_k\} \) \((k \in \mathbb{N})\) be an increasing sequence of rational numbers in \((0, 1)\), and let \( r_0 = 0 \). Denote the interval \((r_{k-1}, r_k)\cap \mathfrak{M}\) by \( I(i, j) \), where \((i, j)\) is the \(k\)th term in the ordering of \( \mathbb{N} \times \mathbb{N} \) defined by \((i, j) < (i', j')\) if and only if \( i+j < i'+j' \), or \( i+j = i'+j' \) and \( j < j' \). We wish to determine the sequence \( \{r_k\} \) in such a way that
\[
a(i, j) < \sum_{n=1}^{j} \mu(I(i, n)) < a(i, j + 1)
\]
for all \( i \) and \( j \). Using the fact that \( \mu([0, x] \cap \mathfrak{M}) \) is a strictly increasing continuous map of \([0, 1]\) onto \([0, \mu(\mathfrak{M})]\), it is easy to see that such a sequence \( \{r_k\} \) can be defined inductively, and that \( r_k \) will necessarily tend to 1. Then the sets \( U_i = \bigcup_{i=j}^{\infty} I(i, j) \) constitute a partition of \( \mathfrak{M} \) with the required properties.

Lemma 2. If \( \mu \) and \( \nu \) are two everywhere positive, nonatomic, Borel measures in \( \mathfrak{M} \), if \( \rho \) is a metric compatible with the topology of \( \mathfrak{M} \), and if \( U \) and \( V \) are open sets such that \( \mu(U) = \nu(V) > 0 \), then for each \( \epsilon > 0 \) there exist partitions \( \{U_i\} \) of \( U \) and \( \{V_i\} \) of \( V \) into nonempty open sets of diameter less than \( \epsilon \) such that \( \mu(U_i) = \nu(V_i) \) for all \( i \in \mathbb{N} \).

Proof. Let \( \{H_i\} \) \((i \in \mathbb{N})\) be a partition of \( V \) into nonempty open sets of diameter less than \( \epsilon \). Since \( U \) is a copy of \( \mathfrak{M} \), by Lemma 1 there exists a partition \( \{G_i\} \) of \( U \) into open sets such that \( \mu(G_i) = \nu(H_i) \) for all \( i \). Let \( \{G_{ij}\} \) \((j \in \mathbb{N})\) be a partition of \( G_i \) into nonempty open sets of diameter less than \( \epsilon \). Since \( H_i \) is a copy of \( \mathfrak{M} \) there exists a partition \( \{H_{ij}\} \) of \( H_i \) into open sets such that \( \mu(G_{ij}) = \nu(H_{ij}) \) for all \( j \). The families \( \{G_{ij}\} \) and \( \{H_{ij}\} \) constitute partitions of \( U \) and \( V \) having the required properties.

Proof of Theorem 1. It suffices to prove the second assertion. Let \( \rho \) be a metric with respect to which \( \mathfrak{M} \) is complete. By repeated application of Lemma 2 we obtain partitions \( U_n = \{U(i_1, \ldots, i_n)\} \) and \( V_n = \{V(i_1, \ldots, i_n)\} \) of \( \mathfrak{M} \) into nonempty open sets of \( \rho \)-diameter less than \( 1/n \) such that \( U(i_1, \ldots, i_n) \supseteq U(i_1, \ldots, i_{n+1}) \), \( V(i_1, \ldots, i_n) \supseteq V(i_1, \ldots, i_{n+1}) \), and \( \mu(U(i_1, \ldots, i_n)) = \lambda(V(i_1, \ldots, i_n)) \) for all \( n \in \mathbb{N} \) and all sets of indices \( i_k \in \mathbb{N} \). For each \( x \in \mathfrak{M} \) there is a unique sequence \( \{i_n\} \) such that \( x \in \bigcap_{n=1}^{\infty} U(i_1, \ldots, i_n) \). Define \( T(x) = \bigcap_{n=1}^{\infty} V(i_1, \ldots, i_n) \). Then \( T \) is a 1-1 map of \( \mathfrak{M} \) onto itself, and
$T(U(i_1, \ldots, i_n)) = V(i_1, \ldots, i_n)$. Since the union of each of the families \( \{U_n\} \) and \( \{V_n\} \) is a base with the property that any open set is the union of some disjoint subclass, it follows that $T$ is a homeomorphism and that $\mu(E) = \lambda(T(E))$ for every Borel set $E$ in $\mathfrak{X}$.

Proof of Theorem 2. Let \( \{x_i\} \) be a countable dense sequence in $X$. Let \( \{r_j\} \) be a sequence of positive real numbers tending to zero such that $\mu(\{x : \rho(x, x_i) = r_j\}) > 0$ for at most countably many values of $r_i$. (Such a sequence exists because for each $i$, $\mu(\{x : \rho(x, x_i) = r\}) > 0$ for at most countably many values of $r$.) Let $S_{ij} = \{x : \rho(x, x_i) = r_j\}$ and $U_{ij} = \{x : \rho(x, x_i) < r_j\}$. Then \( \{U_{ij}\} \) is a base for the topology of $X$. Let $S$ be the union of all the sets $S_{ij}$, and let $G$ be the union of all the sets $U_{ij}$ such that $\mu(U_{ij}) = 0$. Then $\mu(G \cup S) = 0$, the $G_\delta$ set $Y = X - (G \cup S)$ is a topologically complete, separable, 0-dimensional subspace, and $\mu_Y$ is everywhere positive. Let $D$ be a countable dense subset of $Y$, and let $A = G \cup S \cup D$. Then $\mu(A) = 0$, and the $G_\delta$ set $B = Y - D = X - A$ is both dense and frontier in $Y$. Hence $B$ is homeomorphic to $\mathfrak{X}$, and $\mu_B$ is an everywhere positive, nonatomic, normalized Borel measure in $B$. By Theorem 1, $(B, \mu_B)$ is homeomorphic to $(\mathfrak{X}, \lambda)$.

3. Uniqueness of $(\mathfrak{X}, \lambda)$. Let $\mathfrak{F}$ denote the class of topological measure spaces $(X, \mu)$, where $X$ is metrizable, separable, and topologically complete (i.e. a Polish space), and $\mu$ is a nonatomic, normalized Borel measure in $X$. The following theorem shows that $(\mathfrak{X}, \lambda)$ is the only member of this class that is topologically contained in each member of the class.

Theorem 3. If $(X, \mu)$ is a member of $\mathfrak{F}$ that is homeomorphic to a subspace of each member of $\mathfrak{F}$, then $(X, \mu)$ is homeomorphic to $(\mathfrak{X}, \lambda)$.

Proof. By hypothesis, $(X, \mu)$ is homeomorphic to some subspace $(Y, \lambda_Y)$ of $(\mathfrak{X}, \lambda)$. Since $\lambda(Y) = 1$, $Y$ must be a dense subset of $\mathfrak{X}$. It follows that $Y$ is nowhere locally compact, as well as topologically complete, separable, and 0-dimensional. Consequently $Y$, and therefore $X$, is homeomorphic to $\mathfrak{X}$. Since $\lambda_Y$ is everywhere positive, $\mu$ must also be. Therefore $(X, \mu)$ is homeomorphic to $(\mathfrak{X}, \lambda)$, by Theorem 1.

4. Approximation of a Borel set by a Cantor subset. As an application of Theorem 2 we give a new proof of the following theorem, recently proved by Gelbaum [3]. As the referee has pointed out, this theorem is implicitly contained in a result of von Neumann [9, Hilfsatz, p. 577].

Theorem 4. Let $X$ be a complete separable metric space, and let $\mu$ be
a nonatomic Borel measure in $X$. Any Borel set $A$ in $X$ with $0 < \mu(A) < \infty$ is the union of a Cantor set and a set of arbitrarily small measure.

Proof. The formula $\nu(E) = \frac{\mu(E \cap A)}{\mu(A)}$ defines a nonatomic, normalized Borel measure $\nu$ in $X$. By Theorem 2, there exists a set $B$ in $X$ such that $\nu(X - B) = 0$ and $(B, \nu_B)$ is homeomorphic to $(\mathcal{M}, \lambda)$, say by $h$. Then $h(A \cap B)$ is a Borel subset of $\mathcal{M}$ with $\lambda(h(A \cap B)) = 1$. Let $C$ be a compact perfect subset of $h(A \cap B)$ with $\lambda(C) > 1 - \epsilon$. Then $h^{-1}(C) \subseteq A$, $\mu(A - h^{-1}(C)) < \epsilon \mu(A)$, and $h^{-1}(C)$ is compact, perfect, and 0-dimensional, therefore homeomorphic to the Cantor set.


References


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