PROPERTIES OF ABSOLUTE SUMMABILITY MATRICES

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1. Introduction. Let $A$ denote a matrix summability method that maps the complex number sequence $x$ into the sequence $Ax$ whose $n$th term is given by

$$(Ax)_n = \sum_{k \geq 1} a_{nk} x_k.$$ 

If $Ax$ is in $l^1$ whenever $x$ is in $l^1$, then $A$ is called an $l^1$ matrix. An $l^1$ matrix $A$ is said to be sum-preserving if for each $x$ in $l^1$,

$$\sum_{n \geq 1} (Ax)_n = \sum_{k \geq 1} x_k.$$ 

The inverse image of $l^1$ under $A$ is denoted by $l_A$.

In [7, p. 129] Steinhaus proved that if $A$ is a regular (i.e., limit-preserving) matrix, then there is a sequence of 0's and 1's such that $Ax$ is not convergent. It follows immediately that a regular matrix cannot sum every bounded sequence. (Cf. [6].) The principal result of this paper is the analogue of this theorem for $l^1$ matrices.

The author is indebted to H. I. Brown for several helpful comments and observations.

2. The main theorem.

Theorem 1. If $A$ is a sum-preserving $l^1$ matrix and $p > 1$, then $l^p \subseteq l_A$.

By using the characterization of sum-preserving $l^1$ matrices given in [5], we see that this theorem is an immediate corollary to the following assertion.

Theorem 2. If $p > 1$ and $A$ is an $l^1$ matrix such that

$$\lim_{n \to \infty} \sup_{k} \sum_{n \geq 1} |a_{nk}| > 0,$$

then $l^p \subseteq l_A$.

Proof. We may assume that for each $n$, $\lim_k a_{nk} = 0$, for otherwise the conclusion is trivial. Let $\epsilon$ be a positive number such that for infinitely many $k$, $\sum_{n \geq 1} |a_{nk}| \geq 2\epsilon$. We now construct integer sequences $\kappa$ and $\nu$ in the following manner.

Presented to the Society, January 26, 1969; received by the editors March 28, 1969.
Choose \( \kappa(1) \) so that \( \sum_{n \geq 1} |a_{n,\kappa(1)}| \geq 2\varepsilon \), then choose \( \nu(1) \) such that

\[
\sum_{n=1}^{\nu(1)} |a_{n,\kappa(1)}| > \varepsilon \quad \text{and} \quad \sum_{n > \nu(1)} |a_{n,\kappa(1)}| < 2^{-1}.
\]

Having defined \( \kappa(i) \) and \( \nu(i) \) for \( i < m \), we choose \( \kappa(m) \) greater than \( \kappa(m-1) \) such that

\[
\sum_{n=1}^{\nu(m-1)} |a_{n,\kappa(m)}| < 2^{-m} \quad \text{and} \quad \sum_{n \geq 1} |a_{n,\kappa(m)}| \geq 2\varepsilon.
\]

Then choose \( \nu(m) \) greater than \( \nu(m-1) \) and satisfying

\[
\sum_{n=1+\nu(m-1)}^{\nu(m)} |a_{n,\kappa(m)}| > \varepsilon \quad \text{and} \quad \sum_{n \geq 1} |a_{n,\kappa(m)}| < 2^{-m}.
\]

If \( k = \kappa(i) \) define \( x_k = i^{-1} \), otherwise \( x_k = 0 \); obviously \( x \) is in \( l^p \). Defining \( \nu(0) = 0 \), we have

\[
\sum_{n=1}^{\nu(N)} |(Ax)_n| = \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \sum_{i \geq 1} i^{-1} |a_{n,\kappa(i)}| \\
\leq \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \left\{ |a_{n,\kappa(m)}| m^{-1} - \sum_{i \neq m} |a_{n,\kappa(i)}| \right\} \\
> \sum_{m=1}^{N} \varepsilon m^{-1} - \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \sum_{i \neq m} |a_{n,\kappa(i)}|.
\]

Since

\[
\sum_{m \geq 1} \sum_{n=1+\nu(m-1)}^{\nu(m)} \sum_{i < m} |a_{n,\kappa(i)}| < \sum_{m \geq 1} 2^{-m},
\]

and

\[
\sum_{m \geq 1} \sum_{n=1+\nu(m-1)}^{\nu(m)} \sum_{i > m} |a_{n,\kappa(i)}| < \sum_{m \geq 1} 2^{-m-1},
\]

it follows that \( Ax \) is not in \( l^1 \).

3. Further remarks and results about \( l_A \). It is worthwhile noting that Theorem 2 gives a necessary condition for \( A \) to map \( l^p \) into \( l^1 \): viz.,

\[
(2) \quad \lim_k \sum_{n \geq 1} |a_{nk}| = 0.
\]
However, this condition is not sufficient to imply that \( l^p \subseteq l_A \), even for diagonal matrices; e.g., consider diag \( \{ 1 / \log(n + 1) \} \).

Property (2) does yield some information about the summability field \( l_A \), but rather than limiting the size of \( l_A \), (2) implies that \( l_A \) cannot be too small. This statement is put in precise language in the next result, which is easily proved.

**Proposition.** If \( A \) is a matrix such that

\[
(3) \quad \lim \inf_k \sum_{n \geq 1} |a_{nk}| = 0,
\]

then \( l_A \subseteq l^p \).

By an obvious modification of the proof of Theorem 2 we can show that a sum-preserving \( l \)-matrix cannot map every sequence of 0's and 1's into \( l^1 \). This is precisely the \( l \)-analogue of Steinhaus' theorem. In [2] R. C. Buck used the Steinhaus theorem to prove a summability characterization of convergent sequences: viz., the bounded sequence \( x \) is convergent if and only if there exists a regular matrix that sums every subsequence of \( x \). In the light of the preceding remarks one might conjecture that members of \( l^1 \) could be characterized by the existence of a sum-preserving \( l \)-matrix for which \( l_A \) contains every subsequence. This, however, is false.

**Example.** Let \( A \) be the matrix mapping given by

\[
(Ax)_1 = x_1 - x_2 \quad \text{and} \quad (Ax)_n = 2x_n - x_{2n-1} - x_{2n} \quad \text{if} \quad n > 1.
\]

Then \( A \) is clearly a sum-preserving \( l \)-matrix, but if \( x \) is a constant sequence then \( (Ay)_n \equiv 0 \) for every subsequence \( y \) of \( x \).

**References**


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