LIFTING OF TOPOLOGICAL ENTROPY

HARVEY B. KEYNES[1]

1. **Introduction.** In this paper, we will primarily be concerned with the following problem: Suppose the flow (Y, ψ) is a transformation group homomorphic image of the flow (X, ϕ). What can be said about equality of the topological entropy of (X, ϕ) and the topological entropy of (Y, ψ)? In general, one can say nothing, since (Y, ψ) may be the one-point flow. We show in this paper that group extensions and inverse limits do preserve entropy. This enables us to show that certain group-like extensions of minimal algebras (see [2]) also preserve entropy. In particular, it follows that minimal distal flows have zero entropy. We finally show that given $r \geqq 0$, there exists a flow whose entropy is r.

All spaces involved are compact Hausdorff. Unless otherwise stated, we will consider flows (X, ϕ), i.e., ϕ is a homeomorphism onto. However, Lemmas (2.2) thru (2.4), and Theorems (2.6) and (2.11) also hold for semiflows, i.e., continuous surjective maps. The entropy of (X, ϕ) will mean topological entropy and will be denoted by $h(X, \phi)$. All other notations and definitions for entropy will follow [1]. Finally, we will denote that (Y, ψ) is a homomorphic image of (X, ϕ) by $(X, \phi) \xrightarrow{\sim} (Y, \psi)$.

2. **Lifting of entropy.**

(2.1). DEFINITION (CF. [2], [3]). (a) The flow (X, ϕ) is an *isometric extension* of (Y, ψ) if $\pi: (X, \phi) \xrightarrow{\sim} (Y, \psi)$ and there exists a continuous map $\rho: \{(x, x_1) \mid \pi x = \pi x_1\} \to R$ such that:

(1) For every y, ρ induces a metric on $\pi^{-1}y$.

(2) There exists a fixed compact metric space M which is isometric to $\pi^{-1}y$ $(y \in Y)$.

(3) If $x, x_i \in X$ and $\pi x = \pi x_1$, then $\rho(\phi^n x, \phi^n x_1) = \rho(x, x_1)$ for all n.

(b) The flow (X, ϕ) is a *group extension* of (Y, ψ) if there exists a compact Hausdorff topological group G such that:

(1) (G, X) is a left freely-acting transformation group whose action commutes with ϕ.

(2) The orbit transformation group $(X/G, \phi)$ is isomorphic to (Y, ψ).

The following result is basically due to R. Bowen.

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(2.2). LEMMA. *Let K be a compact Hausdorff space, and (X, ϕ), (Y, ψ) flows with $\pi: (X, \phi) \xrightarrow{\sim} (Y, \psi)$. Suppose that:*

(a) *For every $y \in Y$, there exists a homeomorphism g_y of K onto $\pi^{-1}(y)$ (with its subspace topology) such that given η an index of X, there exists γ an index of K for which $g_y \times g_y(\gamma) \subset \eta (y \in Y)$.*

(b) *Given an index η of X, there exists an index λ of X such that if $x, x_1 \in X$ with $\pi x = \pi x_1$ and $(x, x_1) \in \lambda$, then $(\phi^n x, \phi^n x_1) \in \eta$ for all n. Then $h(X, \phi) = h(Y, \psi)$.*

PROOF. We need only show $h(X, \phi) \leqq h(Y, \psi)$. Let $\mathfrak{A}$ be a finite open cover for X, and η a Lebesgue index for $\mathfrak{A}$. Choose an index γ of K as in (a), and an index λ of X as in (b). Then $\bigcup_{i=1}^{s} z_i\gamma = K$ for some integer s. By (b), $\bigcup_{i=1}^{s} g_y(z_i)\lambda = \pi^{-1}y$ $(y \in Y)$. Applying (a), we have that $\phi^j(g_y(z_i)\lambda) \subset \phi^j g_y(z_i)\eta \subset U(y, j, i) \in \mathfrak{A}$ $(y \in Y, j \text{ integer} \geqq 0, 1 \leqq i \leqq s)$.

Fix an integer $m \geqq 0$. For every $y \in Y$, let $A(y, i) = \bigcap_{j=0}^{m-1} \phi^{-j} U(y, j, i)$ $(1 \leqq i \leqq s)$. Then $\bigcup_{i=1}^{s} A(y, i) \supset \pi^{-1}(y)$. Using the filter $\{\pi^{-1}(N) \mid N$ a closed neighborhood of $y\}$, then $\bigcup_{i=1}^{s} A(y, i) \supset \pi^{-1}(M_y)$ for some open neighborhood M_y of y. Choose a finite subset F of Y for which $\bigcup \{M_y \mid y \in F\} = Y$, and consider the open cover $\mathfrak{A}^* = \{M_y \mid y \in F\}$.

Now consider (Y, ψ^m). Let n be an integer $\geqq 0$, and $\mathfrak{A}_n^*$ a minimal subcover for $\mathfrak{A}^* \vee \psi^{-m}\mathfrak{A}^* \vee \psi^{-2m}\mathfrak{A}^* \vee \cdots \vee \psi^{-(n-1)m}\mathfrak{A}^*$. We call an n-tuple $(y_0, \cdots, y_{n-1}) \in F^n$ admissible if $\bigcap_{i=0}^{n-1} \psi^{-im} M_{y_i} \in \mathfrak{A}_n^*$. Choose $x \in X$ and let $y = \pi x$. Choose $(y_0, \cdots, y_{n-1})$ admissible with $y \in \bigcap_{i=0}^{n-1} \psi^{-im} M_{y_i}$. Then

$$
\begin{aligned}
x \in \pi^{-1}(y) \subset \bigcap_{i=0}^{n-1} \pi^{-1}\psi^{-im} M_{y_i} = \bigcap_{i=0}^{n-1} \phi^{-im}\pi^{-1} M_{y_i} &\subset \bigcap_{i=0}^{n-1} \phi^{-im}\left(\bigcup_{k=1}^{s} A(y_i, k)\right) \\
&= \bigcap_{i=0}^{n-1} \bigcup_{k=1}^{s} \bigcap_{j=0}^{m-1} \phi^{-(im+j)} U(y_i, j, k) \\
&= \bigcup_{(k_0, \cdots, k_{n-1})} \bigcap_{i=0}^{n-1} \bigcap_{j=0}^{m-1} \phi^{-(im+j)} U(y_i, j, k_i),
\end{aligned}
$$

where the first union is over n-tuples in $[1, s]^n$. Hence the cover $\mathfrak{A}_n = \{\bigcap_{i=0}^{n-1} \bigcap_{j=0}^{m-1} \phi^{-(im+j)} U(y_i, j, k_i) \mid (y_0, \cdots, y_{n-1})$ admissible, $(k_0, \cdots, k_{n-1}) \in [1, s]^n\}$ is a subcover of $\mathfrak{A} \vee \phi^{-1}\mathfrak{A} \vee \cdots \vee \phi^{-nm+1}\mathfrak{A}$ such that $|\mathfrak{A}_n| = |\mathfrak{A}_n^*| s^n$, independent of m. Now

$$
\begin{aligned}
&1/nm \cdot H(\mathfrak{A} \vee \cdots \vee \phi^{-nm+1}\mathfrak{A}) \\
&\qquad \leqq H(\mathfrak{A}_n)/nm = \log(|\mathfrak{A}_n^*| s^n)/nm \\
&\qquad = H(\mathfrak{A}^* \vee \cdots \vee \psi^{-(n-1)m}\mathfrak{A}^*)/nm + \log s/m,
\end{aligned}
$$

since $\mathfrak{A}_n^*$ is minimal. Letting $n \to \infty$, $h(\mathfrak{A}, \phi) \leqq h(\mathfrak{A}^*, \psi^m)/m + \log s/m$

$\leqq h(Y, \psi^m)/m + \log s/m = h(Y, \psi) + \log s/m$. Letting $m \to \infty$, $h(\mathfrak{A}, \phi) \leqq h(Y, \psi)$. The result follows.

It is easy to see, using the continuity of ρ, that the metric topology on a fibre of an isometric extension is the subspace topology, whence (2.2) can be applied. We also have:

(2.3). LEMMA. *Let (X, ϕ) be a group extension of (Y, ψ) with group G. Then $h(X, \phi) = h(Y, \psi)$.*

PROOF. In this case, each fibre is Gx for some $x \in X$. Since G is freely acting, $\phi_x: G \to G_x$, $g \to gx$ is a homeomorphism $(x \in X)$, and a choice set of X will give the homeomorphisms. As G is a compact group of automorphisms of (X, ϕ) it follows by Ascoli's theorem and minimality properties of the compact-open topology that all the various function uniformities on G will give the original uniformity. Applying this to the uniformity of uniform convergence on G yields (a) of (2.2). For (b), we need only show that given an index η, there is an index γ such that $(x, gx) \in \gamma$ for some x implies $(z, gz) \in \eta$ for all $z \in X$. It is sufficient to show that given a neighborhood U of e, there exists an index γ of X such that $(x, gx) \in \gamma$ for some x implies $g \in U$. If not, there is some open neighborhood V of e such that given any index γ of X, there exists $g_\gamma \notin V$ and $x_\gamma \in X$ for which $(x_\gamma, g_\gamma x_\gamma) \in \gamma$. By compactness, we can assume $x_\gamma \to x$, $g_\gamma \to g$. Then $g \notin V$. However $x = gx$, whence G freely acting implies $g = e$. This is a contradiction. The result follows.

Note that if (Z, ρ) is "in-between" (Y, ψ) and a group extension (X, ϕ), then $h(Z, \rho) = h(Y, \psi)$.

(2.4). LEMMA. *Let $((X_\alpha, \phi_\alpha); \pi_\alpha^\beta)$ be an inverse system of flows (i.e., if $\beta \geqq \alpha$, then $\pi_\alpha^\beta: (X_\beta, \phi_\beta) \xrightarrow{\sim} (X_\alpha, \phi_\alpha)$). Let (X, ϕ) be the inverse limit flow. Then $h(X, \phi) = \sup h(X_\alpha, \phi_\alpha)$.*

PROOF. For every α, let $\pi_\alpha: (X, \phi) \xrightarrow{\sim} (X_\alpha, \phi_\alpha)$ be the induced canonical homomorphism. It then follows that $h(X, \phi) \geqq \sup h(X_\alpha, \phi_\alpha)$. Now let $\mathfrak{A}$ be an open cover for X. Since $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha$ open in X_α open in X_α, α arbitrary$\}$ is a base for X, it follows that $\mathfrak{A}$ is refined by an open cover $\mathfrak{B}$ with elements of the form $\pi_\alpha^{-1} U_\alpha$. By compactness, we can assume $\mathfrak{B} = \{\pi_{\alpha_i}^{-1}(U_{\alpha_i}) \mid i = 1, \cdots, n\}$. Let $\alpha \geqq \alpha_i$, $i = 1, \cdots, n$. Since $\pi_{\alpha_i}^{-1}(U_{\alpha_i}) = \pi_\alpha^{-1}((\pi_{\alpha_i}^\alpha)^{-1}(U_{\alpha_i}))$ $(i = 1, \cdots, n)$, we have that $\mathfrak{B} = \{\pi_\alpha^{-1}(V_i) \mid i = 1, \cdots, n\}$, where V_i open in X_α. Thus, $\mathfrak{B}$ satisfies $\pi_\alpha^{-1}\pi_\alpha \mathfrak{B} = \mathfrak{B}$, and $\pi_\alpha \mathfrak{B}$ is an open cover of X_α. Hence $h(\phi, \mathfrak{A}) \leqq h(\phi, \mathfrak{B}) = h(\phi_\alpha, \pi_\alpha \mathfrak{B}) \leqq h(X_\alpha, \phi_\alpha) \leqq \sup h(X_\alpha, \phi_\alpha)$. Thus, $h(X, \phi) \leqq \sup h(X_\alpha, \phi_\alpha)$, and the proof is completed.

As a corollary, if the inverse system $((X_\alpha, \phi_\alpha); \pi_\alpha^\beta)$ with inverse limit flow (X, ϕ) satisfies $h(X_\alpha, \phi_\alpha)=c$ for every α, then $h(X, \phi)=c$. Moreover, if the system is really a sequence, then $h(X, \phi)=\lim h(X_m, \phi_m)$. We also have:

(2.5). COROLLARY. *Let* $((X_i, \phi_i)\mid i\in I)$ *be a family of flows. Then*

$$h(\times_i X_i, \times_i\phi_i) = \sup\{h(\times_F X_i, \times_F\phi_i) \mid F \text{ a finite subset of } I\}$$
$$\leqq \sup\left\{\sum_F h(X_i, \phi_i) \mid F \text{ a finite subset of } I\right\}.$$

PROOF. The first equality follows from the fact that a product is an inverse system over finite subsets of the indexing set. The latter inequality comes from the obvious assertion that $h(\times_F X_i, \times_F\phi_i) \leqq \sum_F h(X_i, \phi_i)$ when F is finite (whether equality holds here is, as yet, unknown).

Finally, the proof of (2.4) shows that if $((X_\alpha, \phi_\alpha))$ is a family of flows and (X, ϕ) is a flow such that

(1) there exists π_α: $(X, \phi)\rightrightarrows(X_\alpha, \phi_\alpha)$ for all α, and

(2) every open cover of X is refined by a cover of the form $\pi_\alpha^{-1}\mathfrak{A}_\alpha$, where $\mathfrak{A}_\alpha$ is an open cover of X_α for some α, then $h(X, \phi)=\sup h(X_\alpha, \phi_\alpha)$.

We now apply (2.3) and (2.4). The following result is immediate by (transfinite) induction.

(2.6). THEOREM. *Let* (X, ϕ) *be a flow for which there exist an ordinal* ν *and families* $((X_\alpha, \phi_\alpha)\mid \alpha\leqq\nu)$, $((Y_\alpha, \psi_\alpha)\mid\alpha\leqq\nu)$ *of flows such that*:

(1) $(X_\nu, \phi_\nu)=(X, \phi)$.

(2) *If* $\alpha<\nu$, $(Y_{\alpha+1}, \psi_{\alpha+1})$ *is a group extension of* (X_α, ϕ_α).

(3) *If* $\alpha<\nu$, *then* $(Y_{\alpha+1}, \psi_{\alpha+1})\rightrightarrows(X_{\alpha+1}, \phi_{\alpha+1})\rightrightarrows(X_\alpha, \phi_\alpha)$.

(4) *If* $\alpha\leqq\nu$ *is a limit ordinal, then* (X_α, ϕ_α) *is the inverse flow of the inverse system* $((X_\beta, \phi_\alpha)\mid\beta<\alpha)$ *defined by* (3). *Then* $h(X, \phi)=h(X_0, \phi_0)$.

The principal application of (2.6) is to minimal flows. A careful reading of [2, Proposition 4.25] yields:

(2.7). PROPOSITION. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be T-subalgebras of* $\mathfrak{M}$ *such that* $\mathfrak{B}$ *is a nontrivial, quasi-separable, group-like extension of* $\mathfrak{A}$. *Then there exists an ordinal* ν *and collections* $(\mathfrak{B}_\alpha\mid\alpha\leqq\nu)$, $(\mathfrak{P}_\alpha\mid\alpha\leqq\nu)$ *of T-subalgebras such that*:

(1) $\mathfrak{B}_0=\mathfrak{A}$, $\mathfrak{B}_\nu=\mathfrak{B}$.

(2) $\mathfrak{B}_\alpha\subset\mathfrak{B}_\beta$ $(\alpha\leqq\beta\leqq\nu)$.

(3) $\mathfrak{B}_{\alpha+1}$ *is an isometric extension of* $\mathfrak{B}_\alpha$ *and* $\mathfrak{P}_{\alpha+1}$ *is a group extension sion of* $\mathfrak{B}_\alpha$ *with* $\mathfrak{B}_\alpha\subset\mathfrak{B}_{\alpha+1}\subset\mathfrak{P}_{\alpha+1}$ $(\alpha<\nu)$.

(4) *For limit ordinals* $\alpha\leqq\nu$,

$$\mathfrak{B}_\alpha = \{\bigcup\{\mathfrak{B}_\beta \mid \beta<\alpha\}\}, \qquad \mathfrak{P}_\alpha = \{\bigcup\{\mathfrak{P}_\beta\mid\beta<\alpha\}\}.$$

Since T is the integers in our case, every group-like extension is quasi-separable [2, p. 24]. The following result is then an immediate corollary of (2.6), noting that the limit algebras in (2.7) yield inverse limit transformation groups.

(2.8). THEOREM. *Let* (X, ϕ) *and* (Y, ψ) *be minimal flows for which* $X\sim\mathcal{B}$, $Y\sim\mathcal{A}$, *where* $\mathcal{A}$ *and* $\mathcal{B}$ *are T-subalgebras of* $\mathfrak{M}$. *Suppose that* $\mathcal{B}$ *is a group-like extension of* $\mathcal{A}$. *Then* $h(X, \phi)=h(Y, \psi)$.

(2.9). COROLLARY. *Let* (X, ϕ) *be minimal distal. Then* $h(X, \phi)=0$.

PROOF. It is shown in [2] that a minimal distal flow is a group-like extension of the one-point flow.

Note that (2.9) for metric X can be deduced directly from the Furstenberg Structure Theorem [3] by using (2.6).

We now construct flows with arbitrary entropy.

(2.10). LEMMA. *Let* (Y, ψ) *be a flow and* n *a positive integer. Form the flow* (X, ϕ), *where* X *is* n *disjoint copies* (Y, i) *of* Y, $(1\leqq i\leqq n)$, *and* $\phi(y, i)=(y, i+1)$, $1\leqq i\leqq n-1$; $\phi(y, n)=(\psi(y), 1)$. *Then* $h(X, \phi)=n^{-1}h(Y, \psi)$.

PROOF. It is clear that each (Y, i) is ϕ^n-invariant and that the flow $((Y, i), \phi^n)$ is isomorphic to (Y, ψ). Hence $h((Y, i), \phi^n)=h(Y, \psi)$. Moreover, $h(X, \phi^n)=\max_i h((Y, i), \phi^n)=h(Y, \psi)$ by [1, Theorem 4]. Since $nh(X, \phi)=h(X, \phi^n)$ [1, Theorem 2], the result follows.

(2.11). THEOREM. *Let* $r\geqq 0$. *Then there exists a flow* (X, ϕ) *such that* $h(X, \phi)=r$.

PROOF. We can clearly assume $r>0$. Choose a flow (Z, ρ) for which $h(Z, \rho)=c>0$ (e.g., a symbolic flow). If m, n are positive integers, there exists a flow with entropy mc/n by applying (2.10) to (Z, ρ^m). Pick an increasing sequence of rationals $(r_n)_{n\geqq 1}$ such that $r_n c\to r$, and let (X_n, ϕ_n) be a flow with entropy $r_n c$. Let (X_0, ϕ_0) be the trivial flow. Let $\mathfrak{F}$ be the finite subsets of the positive integers directed by subset inclusion. For every $F\in\mathfrak{F}$, let (X_F, ϕ_F) be the disjoint sum flow $+((X_n, \phi_n) \mid n\in F\cup\{0\})$. If $F\leqq G$, define π^G_F: $(X_G, \phi_G)\xrightarrow{\sim}(X_F, \phi_F)$ by $\pi^G_F \mid X_n=\mathrm{id}$, if $n\in F$, and $\pi^G_F\mid X_n$ is the constant map onto X_0, if $n\notin F$. Let (X, ϕ) be the inverse limit flow of this system. Then $h(X, \phi)=\sup\{h(X_F, \phi_F) \mid F\in\mathfrak{F}\}$ by (2.4). But $h(X_F, \phi_F)=\max_F r_n c$ since $(r_n c)\uparrow$. Thus, $h(X, \phi)=\sup_n r_n c=r$. The result follows.

It would be interesting to find a minimal flow which satisfies the conclusion of (2.11). It is known [4] that there exist minimal flows with arbitrarily large entropy.

We finally consider a nonminimal distal flow (X, ϕ). Although X decomposes into a partition of minimal distal flows, we cannot conclude that $h(X, \phi)=0$, since the partition may be infinite. We do know:

(2.12). REMARK. *Let* (X, ϕ) *be a distal flow and* $(E(X), \psi)$ *its enveloping semigroup. Then* $h(E(X), \psi)=0$.

PROOF. Since (X, ϕ) is distal, it is known that $(E(X), \psi)$ is minimal distal. The result follows by (2.8).

The author conjectures that if (X, ϕ) is a distal flow, then $h(X, \phi) =0$.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA AND
UNIVERSITY OF MINNESOTA