COMMUTATIVE QF-1 ARTINIAN RINGS ARE QF

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Abstract. In a recent paper, D. R. Floyd proved several results on algebras, each of whose faithful representations is its own bicommutant (= R. M. Thrall's QF-1 algebras, a generalization of QF-algebras) among which was the theorem in the title for algebras. We obtain our extension of Floyd's result by use of interlacing modules, replacing his arguments involving the representations themselves.

In [10], Thrall observed that the class of finite-dimensional algebras over which every faithful representation has the double centralizer property (i.e., is its own bicommutant) properly contains the class of quasi-Frobenius (= QF) algebras. He called the members of the former class QF-1 algebras and posed the intriguing problem of characterizing these algebras in terms of ideal structure. Solutions for this problem have been given for generalized uniserial algebras [5] and for commutative algebras [4]. Every faithful module M over a QF ring R has the double centralizer property in the sense that the natural homomorphism \( \lambda: R \rightarrow \text{Hom}_C(M, M) \) (where \( C = \text{Hom}_R(M, M) \)) is onto (see [2, §59]). Thus Thrall's definition and his problem extend naturally to QF-1 artinian rings.

In view of recent results on the dominant dimension of an artinian ring (see [1, Theorem 2] or [8, Lemma 9]), the characterization of QF-1 generalized uniserial algebras (and its proof) given in [5] remains valid for generalized uniserial rings. In this note we prove the theorem of the title, thus extending a theorem that Floyd proved for finite-dimensional algebras by means of matrix representations [4].

It is not difficult to show that a direct sum of rings is QF or QF-1 if and only if so is each of the direct summands. Thus for our purposes we may assume that R is a commutative local artinian ring. According to Nakayama's original definition [9, p. 8], such a ring is QF if and only if its R-socle (i.e., its largest semisimple R-submodule) \( S(R) = S(R|R) \) is simple. We shall prove the theorem by constructing, in the event that \( S(R) \) is not simple, a faithful module whose double centralizer has an R-socle larger than that of R. The methods used in this

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construction are suggested by the Lemma of [7] and Theorem 3.1 of [3].

Let \( R \) be a commutative local artinian ring and let \( M \) be a finitely generated indecomposable \( R \)-module with centralizer \( C = \text{Hom}_R(M, M) \) and double centralizer \( C' = \text{Hom}_C(M, M) \). Let \( K = R/\text{Rad} R \) and \( D = C/\text{Rad} C \). In this setting we have the following.

1. The ring \( C \) is completely primary, in the sense that \( D \) is a division ring and \( \text{Rad} C \) is nilpotent [6, Chapter 4].

2. Since \( R \) is commutative, \( C \) and \( C' \) are algebras over \( R \), via

   \[(r\gamma)(m) = r\gamma(m) = \gamma(rm) = (\gamma r)(m)\]

   for \( r \in R, \gamma \in \text{Hom}(M, M), m \in M \).

3. \( (\text{Rad} R) C \) is a nilpotent ideal in \( C \), so \( D \) is an algebra over the field \( K \).

4. The \( C \)-socle of \( M \), \( S(cM) \), is an \( R-C \)-module annihilated by \( \text{Rad} C \) and hence is a \( K-D \)-vector space.

With these observations we can now prove that the \( R \)-socle of \( C', S(rC') \), has length at least as large as the dimension of \( S(cM) \) over \( K \). That is,

\[
|S(rC') : K| \geq |S(cM) : K|.
\]

**Proof.** Let \( T \) be a maximal \( C \)-submodule of \( M \). Then, because the functor \( \text{Hom}_C( , ) \) is left exact in both variables and \( C \) is an \( R \)-algebra, there is an \( R \)-monomorphism

\[
0 \rightarrow \text{Hom}_C(M/T, S(cM)) \rightarrow C'.
\]

But since \( \text{Rad} C \) (and hence \( \text{Rad} R \)) annihilates \( M/T \) and \( S(cM) \) this is really a \( K \)-monomorphism

\[
0 \rightarrow \text{Hom}_D(M/T, S(cM)) \rightarrow S(rC').
\]

Thus, since \( M/T \) is a one-dimensional \( D \)-space and \( S(cM) \) is a finite-dimensional \( D \)-space, we have

\[
|S(rC') : K| \geq |\text{Hom}_D(M/T, S(cM)) : K| = |S(cM) : D| \cdot |D : K| = |S(cM) : K|.
\]

Now we are in a position to prove our main result.

**Theorem.** Every commutative artinian QF-1 ring is QF.

**Proof.** Suppose that \( R \) is as above and has distinct (but necessarily isomorphic) minimal ideals \( S \) and \( S' \). Let \( \phi : S \rightarrow S' \) be an isomorphism and form the interlacing module
\[ M = (R \times R)/L, \quad L = \{(s, -\phi(s)) \mid s \in S\}. \]

Then \( M \) contains a copy of \( R \) and so is faithful. Suppose \( \gamma \) is an \( R \)-endomorphism of \( M \). If \( \eta: R \times R \to M \) is the natural epimorphism then, using the projectivity of \( R \times R \), one obtains an \( R \)-map \( \tilde{\gamma} \) making the diagram

\[
\begin{array}{ccc}
R \times R & \xrightarrow{\gamma} & R \times R \\
\downarrow \eta & & \downarrow \eta \\
M & \xrightarrow{\gamma} & M
\end{array}
\]

commute and consequently taking \( L \) into \( L \). The operation of \( \tilde{\gamma} \) on \( R \times R \) is just that of some matrix

\[
\begin{pmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{pmatrix}, \quad \gamma_{ij} \in R.
\]

The stability of \( L \) under \( \tilde{\gamma} \) yields for each \( s \in S \), an \( \bar{s} \in S \) such that \( s \) and \( \bar{s} \) satisfy the matrix equation

\[
(s, -\phi(s)) \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = (\bar{s}, -\phi(\bar{s})).
\]

From these equations and the independence of \( S \) and \( S' \) it follows easily that \( \gamma_{12} \) and \( \gamma_{21} \) both annihilate \( S \) and so are nilpotent, and that \( \gamma_{11} \) and \( \gamma_{22} \) are simultaneously either nilpotent or invertible. Thus the matrix of \( \tilde{\gamma} \) either has all nilpotent entries or is invertible. But \( \gamma \) is nilpotent or invertible if \( \tilde{\gamma} \) is, by the commutativity of the diagram, so \( M \) is indecomposable. Moreover, if \( \gamma \) is nilpotent then \( \tilde{\gamma} \), having a matrix with radical entries, must annihilate \( S(R) \times S(R) \). That is,

\[
\gamma(\eta(S(R) \times S(R))) = \eta(\tilde{\gamma}(S(R) \times S(R))) = 0
\]

whenever, in our earlier notation, \( \gamma \in \text{Rad } C \). This proves that

\[
\eta(S(R) \times S(R)) \subseteq S(cM)
\]

and the containment is as \( K \)-spaces. Now surely

\[
|\eta(S(R) \times S(R)) : K| = 2 |S(R) : K| - 1,
\]

so by (5) the double centralizer \( C' \) of \( M \) must have an \( R \)-socle strictly larger than that of \( R \). This completes the proof.

If \( R \) is a semisimple ring then all \( R \)-modules (faithful or not) have the double centralizer property. Thus one wonders which rings satisfy this condition. Recalling that a ring is uniserial if and only if each of its factor rings is QF (see [5]), we obtain the
Corollary. Every module over a commutative artinian ring $R$ has the double centralizer property if and only if $R$ is a uniserial ring.

Added in proof. V. P. Camillo has independently obtained these results using different methods.

References


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