EXPANSIVE HOMEOMORPHISMS ON COMPACT MANIFOLDS

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Abstract. In this paper theorems are proved which provide for lifting and projecting expansive homeomorphisms through pseudo-covering mappings so that the lift or projection is also an expansive homeomorphism. Using these techniques it is shown that the compact orientable surface of genus 2 admits an expansive homeomorphism.

1. Introduction. Given a light open map \( \phi: X \to Y \), the branch set \( B_\phi \) is the set at which \( \phi \) fails to be a local homeomorphism. If the restriction of \( \phi \) to \( X - \phi^{-1}(\phi(B_\phi)) \) is a finite-to-one covering map then \( \phi \) will be called a pseudo-covering map (see Definition 5 in [2]). A homeomorphism, \( f \), of a space \( X \) with metric \( d \), will be called expansive (with expansive constant \( c > 0 \)) if to each pair of distinct points \( x, y \) of \( X \) there corresponds an integer \( n \) such that \( d(f^n(x), f^n(y)) > c \).

In [4] theorems are proved for lifting and projecting expansive homeomorphisms through covering maps and for lifting them through pseudo-covering maps. In §2 of this paper a theorem for projecting expansive homeomorphisms through pseudo-covering maps is proved. In §3, Corollary 3.2 provides for the existence of expansive homeomorphisms on 2-manifolds with less stringent conditions than those of Corollary 4.6 of [4]. In §4 it is proved that the compact orientable surface of genus 2 admits an expansive homeomorphism.

2. A projection theorem. In the following \( M \) and \( N \) are compact \( n \)-manifolds with metrics \( r \) and \( d \) respectively. Let \( \phi: M \to N \) be a pseudo-covering map with \( \phi \) one-to-one on \( \phi^{-1}(\phi(B_\phi)) \). In particular \( \phi^{-1}(\phi(B_\phi)) = B_\phi \) and since \( B_\phi \) is closed the restriction of \( \phi \) to \( B_\phi \) is a homeomorphism. Let \( f: M \to M \) be a fibre-preserving homeomorphism such that \( f \) sends \( B_\phi \) onto \( B_\phi \). Thus \( f \) induces a homeomorphism \( g: N \to N \).

Definition 2.1. The map \( f \) is expansive on fibres if there is a number \( e > 0 \) such that for any two points \( x, y \) in \( N \) there exists an integer \( n \) with \( \min r(f^n(z), f^n(w)) > e \) for each pair \( z \in \phi^{-1}(x), w \in \phi^{-1}(y) \).
Theorem 2.2. Let \( \phi: M \to N \) be a pseudo-covering map and \( f: M \to M \) a fibre-preserving homeomorphism with \( f: B_\phi \to B_\phi \). Let \( g: N \to N \) be the homeomorphism induced on \( N \). Then \( g \) is expansive if and only if \( f \) is expansive on fibres.

Proof. Suppose \( f \) is expansive on fibres with corresponding constant \( e \). Let \( x, y \) be in \( N \). We consider three cases.

Case 1. Both \( x \) and \( y \) are in \( \phi(B_\phi) \). We have assumed that \( \phi \) is one-to-one on \( B_\phi \). Thus the restriction of \( g \) to \( \phi(B_\phi) \) can be expressed as \( \phi \circ \phi^{-1} \). Since \( \phi(B_\phi) \) is compact \( \phi^{-1} \) is uniformly continuous on \( \phi(B_\phi) \) and thus Bryant's theorem [1, Theorem 1] applies to prove \( g \) expansive on \( \phi(B_\phi) \) with some expansive constant \( c \).

Case 2. Both \( x \) and \( y \) in \( N - \phi(B_\phi) \). For each \( b \in B_\phi \) choose an open neighborhood \( S_b \) with center \( b \) such that diameter \( S_b < e/3 \) and also such that if \( r(a, b) > e/3 \) then \( \phi(S_a) \cap \phi(S_b) \) is empty. This can be done since the restriction of \( \phi \) to \( B_\phi \) is a homeomorphism. Now for each \( x \in N - \phi(B_\phi) \) choose an elementary neighborhood \( V_x \) for the covering map associated with \( \phi \) such that the components \( U_{x,i} \) of \( \phi^{-1}(V_x) \) have diameter \( < e \). The sets \( \{ \phi(S_b) \} \cup \{ V_x \} \) form an open cover of \( N \). Let \( \beta \) be the Lebesgue number of this cover. Since \( f \) is expansive on fibres there is an \( m \) such that the points \( f^m(z), f^m(w) \) are not both contained in any of the sets \( U_{x,i} \) or \( S_b \) where \( z \) is any point of the fibre over \( x \) and \( w \) is any point of the fibre over \( y \). Also if \( f^m(z) \) is in \( S_a \) and \( f^m(w) \) is in \( S_b \) then \( r(a, b) > e/3 \) and thus \( \phi(S_a) \cap \phi(S_b) \) is empty. It follows that the points \( g^m(x), g^m(y) \) are not both contained in any element of the cover for \( N \) and therefore \( d(g^m(x), g^m(y)) > \beta \).

Case 3. \( x \in \phi(B_\phi) \) and \( y \in N - \phi(B_\phi) \). Let \( \eta \) be greater than 0 and consider the open set \( N(B_\phi, \eta) \) of points whose distance from \( B_\phi \) is less than \( \eta \). The function \( T: B_\phi \times (M - N(B_\phi, \eta)) \to R \) defined by \( T(b, m) = d(\phi(b), \phi(m)) \) is never zero and hence is always greater than some positive number, \( h \). Thus if the fibre over \( g^m(y) \) lies outside of \( N(B_\phi, \eta) \) for some \( m \) then \( d(g^m(x), g^m(y)) > h \). We consider what happens when the fibre over \( y \) remains close to \( B_\phi \). Since the restriction of \( \phi \) to \( B_\phi \) is a homeomorphism there is a \( \delta > 0 \) such that if \( a, b \) are in \( B_\phi \) and \( d(\phi(a), \phi(b)) < \delta \) then \( r(a, b) < e/2 \). Since \( \phi \) is uniformly continuous there is a \( \mu > 0 \) such that \( r(x, y) < \mu \) implies \( d(\phi(x), \phi(y)) < \delta/2 \). Choose \( \eta < \min(e/2, \mu) \). Let \( b = \phi^{-1}(x) \). Since \( f \) is expansive on fibres we can choose \( m \) so that \( r(f^m(b), f^m(w)) > e \) for each \( w \) in \( \phi^{-1}(y) \).

Suppose \( f^m(w) \) is in \( N(B_\phi, \eta) \). Then \( r(a, f^m(w)) < \eta \) for some \( a \in B_\phi \) and \( r(f^m(b), a) > e/2 \). Then \( d(\phi f^m(w), \phi(a)) = d(g^m(y), \phi(a)) < \delta/2 \) and \( d(\phi f^m(b), \phi(a)) = d(g^m(x), \phi(a)) > \delta \) and hence \( d(g^m(x), g^m(y)) > \delta/2 \). Thus \( g \) is expansive with expansive constant \( \min(c, \beta, h, \delta/2) \).
Conversely suppose $f$ is not expansive on fibres. Let $c$ be any constant $>0$. Choose $e>0$ such that $r(u,v)<e$ implies $d(\phi(u),\phi(v))<e$. There exist points $x,y$ in $N$ such that for any $m$ there is a $z\in \phi^{-1}(x)$ and a $w\in \phi^{-1}(y)$ and $r(f^m(z),f^m(w))<e$. Therefore $d(g^m(x),g^m(y))<c$ for all $m$ and $g$ is not expansive.

The following theorem, analogous to Theorem 1 in [1], may be useful for modifying homeomorphisms expansive on fibres. Its proof follows that of Bryant's theorem.

**Theorem 2.3.** If $\phi:M\rightarrow N$ is a pseudo-covering map, $g:M\rightarrow M$ is a fibre-preserving homeomorphism, and $f:M\rightarrow M$ is a homeomorphism which is expansive on fibres, then $g^{-1}fg$ is expansive on fibres.

**Proof.** Let $f$ be expansive on fibres with expansive constant $e$. Using the uniform continuity of $g$, choose $\eta$ so that $r(a,b)<\eta$ implies $r(g(a),g(b))<e$. Now given $x,y$ in $N$ the sets $\{g(z):z\in \phi^{-1}(x)\}$ and $\{g(w):w\in \phi^{-1}(y)\}$ are fibres. Thus there is an $m$ such that $r(f^m g(z),f^m g(w))>e$ for each $z$ and $w$. It follows that

$$r((g^{-1}fg)^m(z),(g^{-1}fg)^m(w))=r((g^{-1}f^m g(z),g^{-1}f^m g(w)))>\eta$$

for each $z$ and $w$ and thus $g^{-1}fg$ is expansive on fibres with expansive constant $\eta$.

**Remark 2.4.** The three-fold iterate of Reddy's torus homeomorphism [5, p. 631] is fibre preserving with respect to a pseudo-covering map, $\phi$ of $S^2$ by the torus. The map $\phi$ is defined by $\phi(x,y,z)=(x^2-y^2,2xy,z)$ with the torus represented as in §4 below. However this homeomorphism is not expansive on fibres and hence the homeomorphism induced on $S^2$ is not expansive.

3. **Lifting expansive homeomorphisms.** Again, $\phi:M\rightarrow N$ will be a pseudo-covering mapping of a compact $n$-manifold with $\phi$ one-to-one on $\phi^{-1}(\phi(B_\phi))$. We will also use $\phi$ for the associated covering map $\phi:M-B_\phi\rightarrow N-\phi(B_\phi)$.

**Theorem 3.1.** Let $g:N\rightarrow N$ be a homeomorphism which sends $\phi(B_\phi)$ onto $\phi(B_\phi)$. Assume that dimension $B_\phi<n-1$. Then $g$ lifts to a homeomorphism $f:M\rightarrow M$ if and only if $g_\phi\pi_1(M-B_\phi,m)$ is conjugate to $\phi_\phi\pi_1(M-B_\phi,m')$ in $\pi_1(N-\phi(B_\phi),g(n))$. Here $m$ is a point in $\phi^{-1}(n)$ and $m'$ is in $\phi^{-1}(g(n))$.

**Proof.** The condition of the theorem is equivalent to the statement that the pairs $(M-B_\phi,\phi)$ and $(M-B_\phi, g\phi)$ are isomorphic covering spaces of $N-\phi(B_\phi)$. Hence the condition implies that $g$ lifts to a homeomorphism $f:M-B_\phi\rightarrow M-B_\phi$ and, by the extension theorem of Fox [3, p. 247], $f$ extends uniquely to a homeomorphism $f:M\rightarrow M$ which covers $g$. 

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In particular, if $g$ is homotopic to the identity by a homotopy which leaves $\phi(B_\phi)$ fixed, then the condition of this theorem is satisfied. This yields the following improvement of Corollary 4.6 in [4].

**Corollary 3.2.** If $g$ is an expansive homeomorphism of the 2-manifold $N$ on itself, $\phi$ is a pseudo-covering map of $M$ onto $N$ and $g$ is homotopic to the identity by a homotopy which leaves the set $\phi(B_\phi)$ pointwise fixed, then $g$ can be lifted to an expansive homeomorphism of $M$.

**Proof.** Since $M$ and $N$ are compact 2-manifolds, $B_\phi$ consists of a finite set of points and thus has dimension 0. The proof then follows from the above observation on Theorem 3.1, and an application of Theorem 3.4 in [4].

4. **Expansive homeomorphism for a surface of genus 2.** In this section we prove the following.

**Theorem 4.1.** The compact orientable surface of genus 2 admits an expansive homeomorphism.

**Proof.** Let $M$ be the surface in question and $N$ the torus. The proof consists in showing that if $g$ is the three-fold iterate of the Reddy torus homeomorphism, then $g$ lifts to $M$ through the pseudo-covering map of $N$ by $M$ described in [4, §5]. Then Theorem 3.4 of [4] completes the proof. It suffices to show that $g$ lifts through the associated covering map.

Represent the torus as the set of points,

$$\{(x, y, z) \in \mathbb{R}^3: ((x^2 + z^2)^{1/2} - 2)^2 + y^2 = 1\}.$$ 

The branch set image then consists of the two points $(0, 0, 3)$ and $(0, 0, -3)$. Choose $(0, 0, 1)$ as basepoint, $m$. Then $\pi_1(M - \phi(B_\phi), m)$ is a free group on three generators $\alpha, \beta,$ and $\gamma$ where $\alpha$ is a loop around the torus in the portion $x>0$, $\beta$ corresponds to the circle $x^2+z^2=1$, and $\gamma$ is represented by a small circle around the point $(0, 0, 3)$. The homeomorphism $g$ is induced by a linear map of the plane. We identify the lower half of the unit square with the back half of the torus, the portion corresponding to $y<0$. Thus $(0, 1/2), (1/2, 1/2)$ correspond to the branch point images $(0, 0, 3), (0, 0, -3)$ respectively. The homeomorphism $g$ leaves the points on the $z$-axis fixed. The image of the fundamental group of $M - B_\phi$ under $\phi_*$ is the subgroup of index 2 generated by $\alpha, \beta, \gamma\alpha\gamma^{-1}, \gamma\beta\gamma^{-1}$, and $\gamma^2$. This set of generators corresponds to the Schreier system consisting of 1 and $\gamma$. Under $g_*$ we have the following:
\[ g^*(\gamma) = \gamma \]
\[ g^*(\alpha) = \alpha \beta \gamma \alpha^2 \beta \gamma \alpha^3 \beta \alpha \gamma \alpha, \]
\[ g^*(\beta) = \alpha \beta \gamma \alpha^2 \beta \gamma \alpha^3 \beta \alpha \gamma \alpha, \]
\[ g^*(\alpha \gamma \gamma^{-1}) = \gamma g^*(\alpha) \gamma^{-1}, \]
\[ g^*(\gamma \beta \gamma) = \gamma g^*(\beta) \gamma^{-1}, \]
\[ g^*(\gamma^2) = \gamma^2. \]

It is easy to check that under \( g \), the generators of the subgroup go into the subgroup and thus \( g \) sends this subgroup onto itself. Therefore \( g \) lifts through the associated covering map and we are finished.

Remark 4.2. If we consider the pseudo-coverings of the torus by surfaces of higher genus this situation does not occur. However it may be possible to show that under \( g \), the subgroup of \( \pi_1(N - \phi B_\ast) \) corresponding to the associated covering map goes into a conjugate subgroup and thus get a lifting.

References


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