OPERATOR-VALUED FEYNMAN INTEGRALS OF CERTAIN
FINITE-DIMENSIONAL FUNCTIONALS

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Abstract. Let $C_0[a, b]$ denote the space of continuous functions $x$ on $[a, b]$ such that $x(a) = 0$. Let $F(x) = f_1(x(t_i)) \cdots f_n(x(t_n))$ where $a = t_0 < t_1 < \cdots < t_n = b$. Recently, Cameron and Storvick defined an operator-valued “Feynman Integral.” In their setting, we give a strong existence theorem as well as an explicit formula for the “Feynman Integral” of functionals $F$ as above under weak restrictions on the $f_j$'s. We also give necessary and sufficient conditions for the operator to be invertible and an explicit formula for the inverse.

1. Introduction and notation. Let $\mathcal{L}(L_2)$ be the space of bounded linear operators on $L_2 = L_2(\mathbb{R}, \mu)$. Let $B[a, b]$ be the space of functions on $[a, b]$ which are continuous except for a finite number of finite jump discontinuities.

We need the following definitions of Cameron and Storvick [2]. (The definitions are not intended to imply the existence of the operators involved. In fact the main theorems of [2] give various conditions on $F$ insuring the existence of these operators.) Let $\psi \in L_2$, $\xi \in (-\infty, \infty)$, and $F$ a functional on $B[a, b]$. For $\lambda > 0$, $I_\lambda(F)$ is the operator on $L_2$ defined by the Wiener integral

$$ (I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx. $$

For $\Re \lambda > 0$, $I^a_\lambda(F)$ is defined to be the operator-valued function of $\lambda$ which agrees with $I_\lambda(F)$ for $\lambda > 0$ and is analytic throughout $\Re \lambda > 0$. For $\Re \lambda > 0$ and any partition $\sigma: a = s_0 < s_1 < \cdots < s_n = b$ the operator $I^{a_\sigma}_\lambda(F)$ is defined by the formula

$$ (I^{a_\sigma}_\lambda(F)\psi)(\xi) = \lambda^{n/2}[(2\pi)^n(s_1 - a) \cdots (s_n - s_{n-1})]^{-1/2} \int_{-\infty}^{\infty} \psi^{(n)}(\xi, v_1, \cdots, v_n) \exp \left( - \sum_{j=1}^{n} \frac{\lambda(v_j - v_{j-1})}{2(s_j - s_{j-1})} \right) dv_1 \cdots dv_n. $$

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where \( v_0 = \xi, f_\sigma(v_0, v_1, \ldots, v_n) = F[z(\sigma, v_0, v_1, \ldots, v_n, \cdot)] \) and
\[
\begin{align*}
z(\sigma, v_0, v_1, \ldots, v_n, s) &= v_j & \text{if } s_j \leq s < s_{j+1}, j = 0, 1, \ldots, n-1 \\
&= v_n & \text{if } s = b.
\end{align*}
\]
(If \( n \) is odd we always choose \( \lambda^{1/2} \) with nonnegative real part.) For \( \Re \lambda > 0 \), the operator \( I_\lambda^{eq}(F) \) is defined by
\[
I_\lambda^{eq}(F) = \lim_{\text{norm } \sigma \to 0} I_\lambda^{eq}(F)
\]
where \( \lim \text{norm } \sigma \to 0 \) means the limit with respect to the weak operator topology on \( L_2 \). In case both \( I_\lambda^{an}(F) \) and \( I_\lambda^{eq}(F) \) exist and agree we will denote their common value by \( I_\lambda(F) \). Finally for \( \lambda = -iq, q \in (-\infty, \infty), q \neq 0 \), the operators \( J_q^{an}(F) \) and \( J_q^{eq}(F) \) are defined by
\[
J_q^{an}(F) = \lim_{p \to 0^+} J_q^{an}(F) \quad \text{and} \quad J_q^{eq}(F) = \lim_{p \to 0^+} J_q^{eq}(F).
\]
Again if both exist and agree we denote their common value by \( J_q(F) \).

Throughout the rest of this paper we assume that \( F \) has the special form given in the abstract, where each \( f_j \) is measurable and satisfies \( \|f_j\|_{\infty} < \infty \). This restriction on the \( f_j \)'s is much weaker than in [1, pp. 333–348] and [5, pp. 177–185] where Cameron's earlier definition of the Feynman and related integrals was employed to study functionals of the same type.

The main theorem below does more than establish the existence of \( J_q(F) \) and give a formula for it. First, it shows that \( J_q(F) \) is the strong operator limit of \( I_\lambda(F) \) rather than just the weak operator limit; secondly, the approach of \( \lambda \) to \(-iq\) is restricted only to the open right half plane and not to the line \( \rho - iq \). We mention also that the existence of \( J_q(F) \) is established for every \( q \neq 0 \). In [2], where more complicated functionals \( F \) are dealt with, the theorems give existence of \( J_q(F) \) for almost every \( q \), but for specific \( q \), one cannot tell whether \( J_q(F) \) exists or not.

We will see below that \( I_\lambda(F) \) and \( J_q(F) \) are compositions of multiplication and convolution operators. In the lemma we study the appropriate convolution operators. We also use the following notation:
\[
(w) \int_{-\infty}^{\infty} f(u, v) du \equiv \lim_{A \to \infty} \int_{-A}^{A} f(u, v) du
\]
which means
\[
\lim_{A \to \infty} \int_{-\infty}^{\infty} (u) \int_{-\infty}^{\infty} f(u, y) du - \int_{A}^{A} f(u, y) du \Bigg| dy = 0.
\]

2. **Lemma.** (a) Let
\[
(U_{q} \psi)(y) = (-iq/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp \left( \frac{q(y - x)^{2}}{2} \right) \psi(x) dx
\]
for \( \psi \in L_{2}, y \in (-\infty, \infty) \) and real \( q \neq 0 \). \( U_{q} \) is a unitary operator on \( L_{2} \) and \( U_{q}^{*} = U_{-q}^{1} = U_{-q} \) where \( U_{q}^{*} \) denotes the adjoint of \( U_{q} \).

(b) Let
\[
(C_{\lambda} \psi)(y) = (\lambda/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp \left( \frac{-\lambda(y - x)^{2}}{2} \right) \psi(x) dx
\]
for \( \psi \in L_{2}, y \in (-\infty, \infty) \) and \( \text{Re} \lambda > 0 \). \( C_{\lambda} \) is in \( \mathcal{L}(L_{2}) \), it is one-to-one, its range is contained in the set of equivalence classes of \( L_{2} \) which contain a continuous function, and \( \| C_{\lambda} \| = 1 \).

**Proof.** (a) The fact that \( U_{q} \) is an isometry in \( \mathcal{L}(L_{2}) \) is shown in Lemma 1 of [2], hence to show \( U_{q} \) unitary, we only need to show that it is onto. Now
\[
(U_{q} \psi)(y) = (-iq/2\pi)^{1/2} \exp (iqy^{2}/2) \mathfrak{F}[\exp(iqx^{2}/2)\psi(x)](qy)
\]
where \( \mathfrak{F} \) denotes the \( L_{2} \)-Fourier transform. Thus to show \( U_{q} \) onto it suffices to show that, given an \( L_{2} \) function \( \phi(y) \), there exists an \( L_{2} \) function \( \psi \) such that
\[
\mathfrak{F}[\exp(iqx^{2}/2)\psi(x)](qy) = (2\pi/-iq)^{1/2} \exp (-iqy^{2}/2)\phi(y).
\]
This follows since the maps \( \psi(x) \to \exp(iqx^{2}/2)\psi(x) \) and \( \mathfrak{F} \) are both onto \( L_{2} \) and the \( q \) appearing in the argument just brings about a change of scale.

Since \( U^{-1} = U^{*} \) holds for any unitary operator, we need simply show \( U_{q}^{*} = U_{-q} \). Now the space \( C_{K} \) of continuous functions on \( (-\infty, \infty) \) with compact support is dense in \( L_{2} \) and so to show \( U_{q}^{*} = U_{-q} \) it suffices to show \( (U_{q} \phi, \phi) = (\psi, U_{-q} \phi) \) for \( \psi, \phi \in C_{K} \). This follows from the Fubini Theorem since the integrand in each case is dominated by the integrable function \( |q|^{1/2} |\psi(x)||\phi(y)| \).

(b) \( C_{\lambda} \) is one-to-one for \( (C_{\lambda} \psi)(y) = 0 \) a.e. implies \( \psi(x) = 0 \) a.e. since the Fourier transform of a convolution is the product of the Fourier transforms and the Fourier transform of \( \exp(-\lambda x^{2}/2) \) never vanishes. For any \( \psi \in L_{2} \), \( (C_{\lambda} \psi)(y) \) is continuous since the convolution of \( L_{2} \) functions is continuous.

In Lemma 1 of [2], Cameron and Storvick show that \( \| C_{\lambda} \| \leq 1 \). Using a comment in [3, p. 1045] one can show quite easily that \( \| C_{\lambda} \| = 1 \). However, the following elementary proof seems more in-
structive. Let \( 1/2 < r < k < 1 \) be given. It suffices to find \( \psi \in L_2 \) such that \( \| \psi \| = 1 \) and \( \| C_\lambda \psi \| > r \). Let \( \sigma^2 = 2 |\lambda|^{-2} \text{Re} \lambda \). Now choose integers \( m \) and \( n \) such that \( (2\pi \sigma^2)^{-1/2} \int_{-m\sigma}^{n\sigma} \exp(-x^2/2\sigma^2)dx > k \) and \( (n-m\sigma)k/n > r \). Let \( \psi(x) = (2n)^{-1/2} \chi(-n,n)(x) \). Then \( \| \psi \| = 1 \), and by use of the formula

\[
\int_{-\infty}^{\infty} \exp \left( \frac{-\lambda(x - \xi)^2}{2} - \frac{\lambda(u - \xi)^2}{2} \right) d\xi = \left( \frac{\pi}{\text{Re} \lambda} \right)^{1/2} \exp \left( \frac{-|\lambda|^2(x - y)^2}{4 \text{Re} \lambda} \right).
\]

We obtain

\[
\| C_\lambda \psi \|^2 = (2\pi \sigma^2)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(u)\psi(x) \exp \left( \frac{-(x - u)^2}{2\sigma^2} \right) dx du
\]

\[
\geq (2\pi \sigma^2)^{-1/2} (2n)^{-1} \int_{-n}^{n} \int_{-n}^{u+n} \exp \left( -s^2/2\sigma^2 \right) ds du
\]

\[
\geq (2n)^{-1} \int_{-n+m\sigma}^{n-m\sigma} \int_{u-n}^{u+n} (2\pi \sigma^2)^{-1/2} \exp \left( -s^2/2\sigma^2 \right) ds du
\]

\[
\geq (2n)^{-1} \int_{-n+m\sigma}^{n-m\sigma} \int_{-n-m\sigma}^{n-m\sigma} (2\pi \sigma^2)^{-1/2} \exp \left( -s^2/2\sigma^2 \right) ds du
\]

\[
\geq (2n - 2m\sigma)k/2n > r > r^2.
\]

The following proposition establishes the existence of the operator \( I_\lambda(F) \) for the class of \( F \)'s we are considering.

PROPOSITION. \( I_\lambda(F) \) exists for all \( \lambda \) such that \( \text{Re} \lambda > 0 \) and is given by

\[
(I_\lambda(F)\psi)(\xi) = \lambda^{n/2} [(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2}
\]

\[
\cdot \int_{-\infty}^{\infty} (n) \cdot \int_{-\infty}^{\infty} \psi(v_n) \cdot f_1(v_1) \cdots f_n(v_n)
\]

\[
\cdot \exp \left( -\sum_{j=1}^{n} \frac{\lambda(v_j - v_{j-1})^2}{2(t_j - t_{j-1})} \right) dv_1 \cdots dv_n
\]

where \( v_0 = \xi \) and \( t_0 = a \).

PROOF. Let \( K_\lambda(F) \) denote the operator defined by the right-hand side of (3). In order to show that \( I_{\lambda}^{\text{eq}}(F) \) exists and equals \( K_\lambda(F) \) it suffices to show that
\[ \lim_{\text{norm } \sigma \to 0} (I_\lambda^\varepsilon(F) \psi, \phi) = (K_\lambda(F) \psi, \phi) \]

for all \( \psi, \phi \in L_2 \) where \( I_\lambda^\varepsilon(F) \) is given by equation (2). This follows since for any partition \( \sigma: a = s_0 < s_1 < \cdots < s_m = b \) with norm \( \sigma \leq \min \{ t_i - a, \cdots, t_n - t_{n-1} \} \) we obtain

\[
(I_\lambda^\varepsilon(F) \psi)(\xi) = \lambda^{n/2} \left[ (2\pi)^n (r_1 - a) \cdots (r_n - r_{n-1}) \right]^{-1/2} \\
\times \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \psi(v_1) \cdot f_1(v_1) \cdots f_n(v_n) \\
\times \exp \left( -\sum_{j=1}^{n} \frac{\lambda(v_j - v_{j-1})^2}{2(r_j - r_{j-1})} \right) dv_1 \cdots dv_n
\]

by carrying out \( m-n \) integrations on the right side of equation (2), where \( v_0 = \xi, r_0 = a \) and \( r_j \) is that \( s_k \) such that \( s_k \leq t_j < s_{k+1} \). Then as norm \( \sigma \to 0 \), \( r_j \to t_j \) and so the result follows from the dominated convergence theorem.

For \( \lambda > 0 \), \( K_\lambda(F) \) agrees with the Wiener integral given in equation (1) and is analytic for \( \text{Re } \lambda > 0 \) [2, p. 533], and so \( I_\lambda^{\varepsilon n}(F) \) exists and equals \( K_\lambda(F) \).

**Theorem.** \( J_q(F) \) exists for all real \( q \neq 0 \) and is given by the right-hand side of (3) where \( \lambda = -iq \) and the integrals are interpreted in the mean. In fact \( J_q(F) \) is the strong operator limit of \( I_\lambda(F) \) as \( \lambda \to -iq \) in the right-half plane.

**Proof.** We will establish that the operator defined by the right-hand side of (3) (with \( \lambda = -iq \)), which we will temporarily denote by \( K_q(F) \), is the strong operator limit of \( I_\lambda(F) \) as \( \lambda \to -iq \) in the right half plane; the first statement then follows. Careful examination of \( I_\lambda(F) \) reveals that it is the composition of multiplication operators and convolution operators [2, p. 535]; i.e. \( I_\lambda(F) = C_{1,\lambda} \circ M_1 \circ \cdots \circ C_{n,\lambda} \circ M_n \) where \( C_{j,\lambda} \) and \( M_j \) are the elements of \( \mathcal{L}(L_2) \) defined respectively by

\[
(C_{j,\lambda} \psi)(y) = \left( \frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( \frac{-\lambda(y - x)^2}{2(t_j - t_{j-1})} \right) \psi(x) \, dx
\]

and \( (M_j \psi)(y) = \psi(y)f_j(y) \). Similarly \( K_q(F) = U_1 \circ M_1 \circ \cdots \circ U_n \circ M_n \) where

\[
(U_j \psi)(y) = \left( \frac{-iq}{2\pi(t_j - t_{j-1})} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( \frac{iq(y - x)^2}{2(t_j - t_{j-1})} \right) \psi(x) \, dx.
\]
Now the map \((A, B) \mapsto A \circ B\) of \(L_2 \times L_2\) to \(L_2\) is continuous in the strong operator topology as long as \(A\) is restricted to lie in a bounded subset of \(\mathcal{L}(L_2)\) \([3, \text{p. 512}]\). Since \(\|C_{\lambda}\| = 1\) it will suffice to show that \(C_{\lambda, \wedge} \mapsto U_\wedge\) where the s.o. refers to the fact that the convergence is with respect to the strong operator topology. This suffices since if \(C_{\lambda, \wedge} \mapsto U_n\) then \(C_{\lambda, \wedge} \circ M_n \mapsto U_n \circ M_n\) and \(M_{n-1} \circ C_{\lambda, \wedge} \circ M_n \mapsto M_{n-1} \circ U_n \circ M_n\) and \(C_{\lambda, \wedge} \circ M_{n-1} \circ C_{\lambda, \wedge} \circ M_{n-1} \circ U_n \circ M_n\), etc. In showing \(C_{\lambda, \wedge} \mapsto U_\wedge\) it clearly suffices to consider the case where \(t_j - t_{j-1} = 1\). Hence, in our notation, \(U_j = U_q\) and \(C_{\lambda, \wedge} = C_{\lambda}\). Now to show \(C_{\lambda} \mapsto U_q\) (s.o.), it suffices to show \(\|C_{\lambda} \psi - U_q \psi\| \to 0\) for \(\psi \in C_\kappa\) since for \(\psi_0 \in L_2\) we have the inequalities

\[
\|C_{\lambda} \psi_0 - U_q \psi_0\| \leq \|C_{\lambda} - U_q\| \|\psi_0 - \psi\| + \|C_{\lambda} \psi - U_q \psi_0\|
\]

\[
\leq 2\|\psi_0 - \psi\| + \|C_{\lambda} \psi - U_q \psi\|.
\]

Now to show \(\|C_{\lambda} \psi - U_q \psi\| \to 0\), it suffices \([4, \text{p. 11}]\) to show (a) \(C_{\lambda} \psi \mapsto U_q \psi\) weakly and (b) \(\|C_{\lambda} \psi\| \to \|U_q \psi\|\). To obtain (a), it suffices to show \((C_{\lambda} \psi, \phi) \to (U_q \psi, \phi)\) for \(\phi \in C_\kappa\); one may see this by an inequality similar to (4). Since \((C_{\lambda} \psi, \phi)\) is a complex-valued function, it suffices to show \((C_{\lambda} \psi, \phi) \to (U_q \psi, \phi)\) along any sequence \(\lambda_m \to -iq\) with \(\text{Re} \lambda_m > 0\). But as \(\psi, \phi \in C_\kappa\), \(\|\psi \phi\|\) is integrable over \((-\infty, \infty) \times (-\infty, \infty)\) and so the result follows upon application of the Fubini Theorem and the dominated convergence theorem.

To establish (b) it again suffices to show \(\|C_{\lambda_m} \psi\| \to \|U_q \psi\|\) for any sequence \(\lambda_m \to -iq\) with \(\text{Re} \lambda_m > 0\). Now since \(\|C_{\lambda_m}\| = 1\) and \(U_q\) is an isometry, we have \(\lim \inf \|C_{\lambda_m} \psi\| \leq \lim \sup \|C_{\lambda_m} \psi\| \leq \|\psi\| = \|U_q \psi\|\). But also, since balls about zero in \(L_2\) are compact in the weak topology, only finitely many of the \(C_{\lambda_m}\) can be in any such ball of radius less than \(\|U_q \psi\|\). Hence \(\|U_q \psi\| \leq \lim \inf \|C_{\lambda_m} \psi\|\) and so \(\lim \|C_{\lambda_m} \psi\| = \|U_q \psi\|\) which completes the proof of the theorem.

**Corollary.** \(J_q(F)\) is invertible as an element in \(\mathcal{L}(L_2)\) if and only if each \(f_j\) is bounded away from zero a.e. In this case \(J_q(F)^{-1}\) is given by the formula

\[
(J_q(F)^{-1}\psi)(\xi) = (iq)^{n/2}[(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2}
\]

\[
\cdot \int_{-\infty}^{\infty} (v_{n-1}) \int_{-\infty}^{\infty} \psi(v_n)
\]

\[
\cdot [f_n(\xi)f_{n-1}(v_1)f_{n-2}(v_2) \cdots f_1(v_{n-1})]^{-1}
\]

\[
\cdot \exp \left(\frac{-iq}{2} \sum_{j=1}^{n} \frac{(v_{n-j+1} - v_{n-j})^2}{(t_j - t_{j-1})}\right) dv_n \cdots dv_1
\]

where \(v_0 = \xi\) and \(t_0 = a\).
Proof. It is easily verified [4, p. 32] that $M_j$ is invertible if and only if $f_j$ is bounded away from zero a.e., and, in this case, $M_j^{-1}$ is multiplication by $1/f_j$. Thus under the conditions of the corollary, it follows from the lemma and theorem above that $J_q(F)^{-1} = M_n^{-1} \circ U_n^{-1} \circ \cdots \circ M_1^{-1} \circ U_1^{-1}$ exists and is given by (5). If $J_q(F)$ is not invertible, some $M_j$ must fail to be invertible by the lemma and so $f_j$ is not bounded away from zero a.e.

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Bibliography


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