A SHORT PROOF OF CURRY'S NORMAL FORM THEOREM

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In Chapter 6 of their book [1] Curry and Feys define a notion of reduction (strong reduction) for the extensional theory of equality in combinatory logic, show [1, Theorem 3, p. 221] that strong reduction has the Church-Rosser property, and define a notion of normal form in analogy with the corresponding concept in lambda-conversion. Curry's normal form theorem [1, Theorem 7, p. 230] asserts that if a term ("ob") of combinatory logic is in normal form, it is irreducible, so that if X has normal form X*, then X reduces to X* by a process (namely, strong reduction) that cannot be continued further.

Curry's proof of his theorem in [1] is quite long and difficult (see [3, p. 228] for comment). There is another lengthy proof in the current draft of [2] and the first author has discovered a proof using his axiomatization of strong reduction [3]. The present proof follows the same general line as the latter proof, but it is considerably shorter and simpler.

Definitions and notation are as in [3], except for the symbol > which is used here for weak reduction.

1. Substitution and abstraction. The first result is essentially from [4, Lemmas 1 and 4]. The proofs, by induction, are easy.

Lemma 1. Let P be a redex scheme. Then:
(a) P contains at most one occurrence of each meta-variable;
(b) If M is a meta-variable occurring in P, there is an N such that NM occurs in P;
(c) If P is not basic (i.e. is the result of at least one application of scheme (viii) of [3, p. 233]), then P is weakly irreducible;
(d) If P is not basic, then either \( P \Rightarrow SP_1P_2 \) or \( P \Rightarrow SP_1 \) where \( P_1 \) contains at least one occurrence of an atomic combinator.

The hypotheses of the next lemma are, in essence, the properties of redex schemes asserted in Lemma 1(a), (b), and (c).
Lemma 2. Suppose that \( P \) is weakly irreducible, that \( P \) contains at most one occurrence of each meta-variable, and that either \( P \) is itself a meta-variable, or the occurrence of any meta-variable \( M \) in \( P \) is in a component \( NM \) of \( P \). If \([X/A][Y/B][Z/C]P \rightarrow R\), then \( R = [X°/A] \cdot [Y°/B][Z°/C]P\), where \( X \rightarrow X°\), \( Y \rightarrow Y°\), and \( Z \rightarrow Z°\).

Proof. The proof is by induction on \( P \), with three basic clauses: \( P \equiv M; P \equiv NM \), where \( N \) does not contain any meta-variables; and \( P \) does not contain any meta-variables. The induction step, \( P \equiv P_1P_2 \), uses the fact that no substitution instance of \( P \) is itself a weak redex.

The next result is Lemma 11 of [3].

Lemma 3. Let \( P \) be a redex scheme and suppose that \( U, V, W \) do not contain \( x \). Then \([U/A][V/B][W/C][x]P_{ijk} = [x][U/A][V/B] \cdot [W/C]P_{ijk} = [x][U*/A][V//B][W//C]P\).

The final preliminary result is an immediate consequence of the Church-Rosser Theorem for weak reduction (see, e.g. [2], or [5, Theorem 12]). It can also be proved directly by induction on \( Y \).

Lemma 4. If \( Y \) is weakly irreducible, \( X = [x]Y \), and \( Xx \rightarrow Q \), then \( Q \rightarrow Y \).

2. The normal form theorem. By [3], \( X \) is irreducible if it contains no redexes, so Curry's theorem may be stated as follows.

Normal form theorem. If \( X \) is in normal form, it contains no redexes.

Proof. The proof is by induction on the definition of normal form.

(1) If \( X = x \), then \( X \) contains no redex by Lemma 1(b) or (d).

(2) If \( X = xX_1 \cdots X_n \), with \( X_1, \ldots, X_n \) in normal form, assume by induction that \( X_1, \ldots, X_n \) contain no redexes. The only possible redexes in \( X \) then have \( x \) at the head, which is impossible by Lemma 1(d).

(3) If \( X = [x]Y \), where \( Y \) is in normal form, assume by induction that \( Y \) contains no redexes. The rest of the proof is by induction on \( Y \). Note that \( Y \) is weakly irreducible.

(3a) If \( Y = x \), then \( X = I \), which is not a redex by Lemma 1(d).

(3b) If \( Y \) does not contain \( x \), then \( X = KY \). By Lemma 1(d), neither \( X \) nor \( K \) is a redex, so any redexes in \( X \) must be components of \( Y \).

(3c) If \( Y = Xx \) and contains no redexes, clearly \( X \) contains none.

(3d) If \( Y = Y_1Y_2 \) and \( X = SX_1X_2 \) where \( X_i = [x]Y_i, i = 1, 2 \), then assume the theorem for \( X_1 \) and \( X_2 \). Then the only possible redexes in
X (by Lemma 1(d)) are SX₁ and X itself. Consider the two cases separately.

(3d1) Suppose SX₁ is a redex. Then SX₁ is a substitution instance of a redex scheme SR. If SR is basic, then SR = S(KI), so KI = [x]Y₁. Hence Y₁ ≡ I, contradicting the hypothesis that Y is weakly irreducible. Thus, SR = [x]Pᵢⱼₖ for a redex scheme P. This implies that P ≡ SRM for a meta-variable M, so that X itself is also a redex. Hence, this case may be included in the next.

(3d2) Suppose X ≡ SX₁X₂ is a redex, a substitution instance of a redex scheme SR₁R₂. If SR₁R₂ is basic, then R₁ ≡ KA, so X₁ ≡ KU ≡ [x]Y₁. Then Y₁ ≡ U and does not contain x. Either R₂ ≡ I or R₂ ≡ KB. In the first case, X₂ ≡ I, so Y₂ ≡ x. But then X ≡ [x]Ux ≡ U, which is impossible. In the second case, X₂ ≡ KV and Y₂ ≡ V so that Y does not contain x, and X ≡ [x]Y ≡ KY, contrary to hypothesis. Thus, SR₁R₂ is not basic.

We may now suppose that SR₁R₂ = [x]Pᵢⱼₖ for a redex scheme P, so that X ≡ [U/A][V/B][W/C][x]Pᵢⱼₖ. Since X ≡ [x]Y, X does not contain x, and hence, neither do U, V, W. Thus, Lemma 3 applies and X ≡ [x]Q, where Q ≡ [Uᵢ/A][Vᵢ/B][Wᵢ/C]P. Moreover, we must have Q = Q₁Q₂ and X = S([x]Q₁)([x]Q₂), for the alternative is that Q = S(U₁V₁W₁X₁ ≡ SU₁V₁X₁, so that SR₁R₂ = SAB, which is not a redex scheme. Then by Lemma 4, Q ⊳ Y, Q₁ ⊳ Y₁, and Q₂ ⊳ Y₂.

If P is weakly irreducible, then Lemma 2 (with Y ⊳ R) implies that Y ≡ [Uᵢ₀/A][Vᵢ₀/B][Wᵢ₀/C]P. This contradicts the hypothesis that Y is not a redex.

If P is weakly reducible, then either P = SABC or P = KAB, so that either Q = SUᵢVᵢWᵢ or Q = KuᵢVᵢ. But since Q₁ ⊳ Y₁, either SUᵢVᵢ ⊳ Y₁ and hence Y₁ ≡ SY₁'Y₁, or else Kuᵢ ⊳ Y₁, and hence Y₁ ≡ KY₁'. In either case, Y is not weakly irreducible. This final contradiction completes the proof.

References


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