AN \textit{m-ORTHOCOMPLETE ORTHOMODULAR LATTICE IS m-COMPLETE}

SAMUEL S. HOLLAND, JR.

\textbf{Abstract.} We call an orthomodular lattice $\mathcal{L}$ \textit{m-orthocomplete} for an infinite cardinal $m$ if every orthogonal family of $\leq m$ elements from $\mathcal{L}$ has a join in $\mathcal{L}$, and we call $\mathcal{L}$ \textit{m-complete} if every family, orthogonal or not, of $\leq m$ elements from $\mathcal{L}$ has a join in $\mathcal{L}$. We prove that an $m$-orthocomplete orthomodular lattice is $m$-complete. Since a Boolean algebra is a distributive orthomodular lattice, we obtain as a special case the Smith-Tarski theorem: An $m$-orthocomplete Boolean algebra is $m$-complete.

We refer the reader to [1] for the elementary theory and basic nomenclature of orthomodular lattices, mentioning specifically here only these notational conventions: we write $a - b$ for $a \land b^\perp$ when $b \leq a$, and write $\bigoplus a_\alpha$ for $\bigvee a_\alpha$ when $\alpha \neq \beta \Rightarrow a_\alpha \perp a_\beta$.

\textbf{Lemma.} Let $\mathcal{L}$ be an $m$-orthocomplete orthomodular lattice, $\sigma$ an ordinal number satisfying $\text{card}(\sigma) \leq m$, and $(y_\alpha; \alpha < \sigma)$ a family of elements from $\mathcal{L}$ satisfying

(i) $y_0 = 0$,

(ii) $\alpha \leq \beta < \sigma \Rightarrow y_\alpha \leq y_\beta$ (increasing),

(iii) $\beta$ a limit ordinal $< \sigma \Rightarrow \bigvee (y_\alpha; \alpha < \beta)$ exists and $= y_\beta$ (continuous from the left).

Then for every ordinal $\beta$ satisfying $2 \leq \beta < \sigma$ we have

$$\bigvee (y_\alpha; \alpha < \beta) = \bigoplus (y_{\rho+1} - y_\rho; \rho + 1 < \beta).$$

\textbf{Proof of the Lemma.} Both joins displayed in the assertion of the lemma exist, the orthogonal join by $m$-orthocompleteness, and the other by assumption (iii). (Assumption (iii) covers the case when $\beta$ is a limit ordinal; if $\beta$ is not a limit ordinal, then obviously $\bigvee (y_\alpha; \alpha < \beta) = y_{\beta - 1}$. If $\rho + 1 < \beta$, then $y_{\rho+1} - y_\rho \leq y_{\rho+1} \leq \bigvee (y_\alpha; \alpha < \beta)$; hence

$$\bigoplus (y_{\rho+1} - y_\rho; \rho + 1 < \beta) \leq \bigvee (y_\alpha; \alpha < \beta).$$

We need therefore prove only the statement $P(\beta): \bigvee (y_\alpha; \alpha < \beta) \leq \bigoplus (y_{\rho+1} - y_\rho; \rho + 1 < \beta)$. $P(2)$ is the assertion $y_1 \leq y_1 - y_0$ which is true because $y_0 = 0$. We use transfinite induction. Assume that $P(\gamma)$

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is true for all $\gamma < \beta$. If $\beta$ is a limit ordinal, then for any $\alpha < \beta$, $\alpha + 1 < \beta$ and then, using the induction hypothesis,

$$y_\alpha = \bigvee (y_{\sigma}; \sigma \leq \alpha) = \bigvee (y_{\sigma}; \sigma < \alpha + 1)$$

$$\leq \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \alpha + 1) \leq \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta).$$

Hence $\bigvee (y_\alpha; \alpha < \beta) \leq \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta)$. If $\beta$ is not a limit ordinal, then $\bigvee (y_\alpha; \alpha < \beta) = \bigvee (y_\alpha; \alpha \leq \beta - 1) = y_{\beta - 1}$. Now there are two possibilities: either $\beta - 1$ is a limit ordinal or it is not. If $\beta - 1$ is a limit ordinal, then by (iii) and the induction hypothesis,

$$y_{\beta - 1} = \bigvee (y_\alpha; \alpha < \beta - 1) \leq \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta - 1)$$

$$\leq \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta)$$

and we are done. If $\beta - 1$ is not a limit ordinal, then

$$\bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta)$$

$$= \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 \leq \beta - 1)$$

$$= (y_{\beta - 1} - y_{\beta - 2}) \bigoplus \bigoplus (y_{\rho+1} - y_{\rho}; \rho + 1 < \beta - 1)$$

$$\leq (y_{\beta - 1} - y_{\beta - 2}) \bigoplus \bigvee (y_{\alpha}; \alpha < \beta - 1)$$

$$= (y_{\beta - 1} - y_{\beta - 2}) \bigoplus y_{\beta - 2} = y_{\beta - 1},$$

which proves $P(\beta)$. (In the second to the last step we used the induction hypothesis.)

**Theorem.** An $m$-orthocomplete orthomodular lattice is $m$-complete.

**Proof.** By induction. Let $(x_\gamma; \gamma \in \Sigma)$ be a family of elements from $\mathcal{L}$ indexed by a set $\Sigma$ with $\text{card}(\Sigma) \leq m$, and assume that the join of any $\Sigma'$-indexed family exists when $\text{card}(\Sigma') < \text{card}(\Sigma)$. Let $\sigma$ be the least ordinal corresponding to $\text{card}(\Sigma)$. We can suppose that $\text{card}(\Sigma)$ is infinite so that $\sigma$ is a limit ordinal, and we can suppose that we have replaced the set $\Sigma$ by the set $(\alpha; \alpha < \sigma)$ so that we are dealing with an ordinal-indexed family $(x_\alpha; \alpha < \sigma)$. By the induction assumption $y_\alpha = \bigvee (x_\rho; \rho < \alpha)$ exists for every $\alpha < \sigma$. This family $(y_\alpha; \alpha < \sigma)$ satisfies the conditions of the lemma, (i) and (ii) being obviously met, and (iii) being a consequence of the following direct computation for $\beta$ a limit ordinal $< \sigma$: $\bigvee (y_\alpha; \alpha < \beta) = \bigvee (x_\rho; \rho < \alpha) = \bigvee (x_\rho; \rho < \beta) = y_\beta$.

The orthogonal join $z = \bigoplus (y_{\alpha+1} - y_\alpha; \alpha + 1 < \sigma)$ exists by $m$-orthocompleteness; this element $z$ is the desired join, $\bigvee (x_\rho; \rho < \sigma)$.

First, note that if $z$ is in fact an upper bound of the set $(x_\rho; \rho < \sigma)$, then, among all such upper bounds, it is certainly the least. For if $w \geq x_\rho$ for all $\rho < \sigma$, then $w \geq \bigvee (x_\rho; \rho < \alpha + 1) = y_{\alpha+1} \geq y_{\alpha+1} - y_\alpha$ for all
α + 1 < σ so \( w \geq z \). Hence it is enough to show that \( z \geq x_β \) for every \( β < σ \).

If \( β < σ \) then, \( σ \) being a limit ordinal, we have \( β + 2 < σ \), whence

\[
x_β \leq \bigvee (x_ρ; ρ < β + 1) = y_{β+1} = \bigvee (y_α; α \leq β + 1)
= \bigvee (y_α; α < β + 2) = \bigoplus (y_{ρ+1} - y_ρ; ρ + 1 < β + 2) \leq z,
\]

where, in the second-to-the-last step, we have used the lemma. That proves the theorem.

Call an orthomodular lattice \( \mathcal{L} \) orthocomplete if it is \( m \)-orthocomplete for every \( m \) (or for \( m = \text{card}(\mathcal{L}) \) which is enough).

**Corollary 1.** An orthocomplete orthomodular lattice is complete.

An orthomodular lattice \( \mathcal{L} \) satisfies the "\( m \)-chain condition" (I am adapting this nomenclature from Sikorski [2, p. 72]) provided that any orthogonal family in \( \mathcal{L} \) has \( \leq m \) nonzero elements.

**Corollary 2.** An \( m \)-orthocomplete orthomodular lattice satisfying the \( m \)-chain condition is complete.

For \( m = \aleph_0 \), this was proved by Zierler [3, Lemmas 1.8 and 1.9].

**Corollary 3 (Smith-Tarski; see [2; §20.1]).** An \( m \)-orthocomplete Boolean algebra is \( m \)-complete.

**Corollary 4 (Tarski; see [2; §20.5]).** An \( m \)-orthocomplete Boolean algebra satisfying the \( m \)-chain condition is complete.

**References**


University of Massachusetts, Amherst, Massachusetts 01003