ON SLOW VARIATION

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A positive function $L$ on the positive real line is said to be "slowly varying at infinity" if, for each $t > 0$:

$$
\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad t \in (0, \infty).
$$

Karamata [3] has proved that if $L$ is continuous, then

$$
L(x) = a(x) \exp \left( \int_1^x \frac{\epsilon(y)}{y} \, dy \right)
$$

where $\epsilon(x) \to 0$ and $a(x) \to c \in (0, \infty)$ as $x \to +\infty$. Feller [2, pp. 272–274] gives a new exposition of the theory and a proof of (1), implicitly assuming not the continuity, but the local integrability of $L$ on some half line $(A, \infty)$. But it has been already proved [1], [4] that measurability of $L$ is enough.

From (1), it follows that [2, footnote p. 302]:

$$
\lim_{x \to +\infty} x^\alpha L(x) = \infty, \quad \lim_{x \to +\infty} x^{-\alpha} L(x) = 0 \quad (\alpha > 0).
$$

The aim of this note is to show that $L$ measurable implies that $L$ is locally bounded on some half line $(A, \infty)$ (thus preparing for Feller's exposition) and to give a short proof of (2) which avoids an appeal to (1), by establishing the following theorem:

**Theorem.** If $L$ is slowly varying and measurable, then for every $\alpha > 0$, there exists $X(\alpha)$ and $T(\alpha)$ such that $x > X(\alpha)$ and $t > T(\alpha)$ imply:

$$
t^{-\alpha} \leq L(tx)/L(x) \leq t^\alpha.
$$

**Proof.** Let $S_n = \{t > 1: t^{-\alpha} \leq L(tx)/L(x) \leq t^\alpha \forall x > n\}$.

From slow variation it follows:

$$
\bigcup_{n=1}^{\infty} S_n = \{t: t > 1\}.
$$

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1 The author is indebted to the referee for these references.
Since $L$ is measurable, there exists $n_0$ such that $S_{n_0}$ has a positive Lebesgue measure. Now $S_nS_n \subseteq S_n$, i.e. $S_n$ is a multiplicative semi-group. Hence the interior of $S_{n_0}$ is not empty; this implies that $S_{n_0}$ contains a half-line $(T(\alpha), \infty)$, and we can take $X(\alpha) = n_0$.

The idea of the proof can be used [5] to get uniform convergence of $L(xt)/L(x)$ on compact subsets of $R^+$. 

**Bibliography**


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