POLYNOMIAL APPROXIMATION ON $y=x^\alpha$

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ABSTRACT. We show that on the curve $y=x^\alpha$, α any irrational, $0\leqq x\leqq 1$, the degree of approximation by nth degree polynomials in x and y in the L^2 norm has order of magnitude $1/n^{3/2}$.

Let Q be a continuous, rectifiable curve, $y=g(x)$, $0\leqq x\leqq 1$. By the L^2 norm of $f(x, y)$ on Q we mean

$$\|f\|_{L^2} = \left(\int_0^1 |f(x, g(x))|^2 dx\right)^{1/2}.$$

The set of contractions, $K(Q)$, is defined as follows: $f(x, y)$ is in $K(Q)$ iff $\omega_f(\delta)\leqq\delta$ for all $\delta>0$, where $\omega_f(\delta)$ is the L^2 modulus of continuity of $f(x, y)$ on Q. Let P_n be the set of all polynomials, $\sum_{i,j=0}^n a_{ij}x^iy^j$. The degree of approximation, $\rho_n(Q)$ is defined as

$$\rho_n(Q) = \sup_{f\in K(Q)} \inf_{p\in P_n} \|f-p\|_{L^2}.$$

In this note we are concerned with the order of magnitude of $\rho_n(Q_\alpha)$, where Q_α is the curve $y=x^\alpha$, α real irrational, $0\leqq x\leqq 1$.

From general considerations it is known that there exist c_1, $c_2>0$ such that

$$c_1/n^2 < \rho_n(Q) < c_2/n, \tag{1}$$

where Q is any continuous rectifiable curve; cf. [1]. It has been shown [1] that on Q_r, r rational, that there exist c_1, $c_2>0$ such that $c_1/n < \rho_n(Q_r) < c_2/n$. For irrational α the results have been rather special. Specifically, if α is an irrational with the property that there exist integers b, n such that $|\alpha-b/n| < n^{-4}(3/(1-\delta))^{-2n^2}$ for infinitely many n, $0<\delta<1$, then there exists a subsequence n_i and c_1, $c_2>0$ such that

$$c_1/n_i^{3/2} < \rho_{n_i}(Q_\alpha) < c_2/n_i^{3/2}. \quad \text{Cf. [1].}$$

We generalize this last result in the following theorem, whose proof is somewhat simpler than that of the above.

THEOREM. *Let α be any real irrational number. Then there exist c_1, $c_2>0$ such that*

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$$c_1/n^{3/2} < \rho_n(Q_\alpha) < c_2/n^{3/2}.$$

PROOF. Let

$$\rho_\Lambda = \sup_{f\in K(0,1)} \inf_{c_\Lambda} \left\| f(x) - \sum_{\lambda\in\Lambda} c_\lambda x^\lambda \right\|,$$

where $K(0, 1)$ consists of the contractions on $[0, 1]$, and Λ is a finite set of positive numbers. Now

$$\rho_n(Q_\alpha) = \sup_{f\in K(Q)} \inf_{p\in P_n} \|f(x, x^\alpha) - p(x, x^\alpha)\|.$$

Let $f^*(x) = f(x, x^\alpha)$. Then $f^*(x)$ is in $K(0, 1)$. Thus

$$\rho_n(Q_\alpha) = \sup_{f^*\in K(0,1)} \inf_{\{a_{ij}\}} \left\| f^*(x) - \sum_{i,j=0}^{n} a_{ij} x^{i+\alpha j} \right\| = \rho_\Lambda$$

where $\Lambda = \{i+\alpha j\}$, $i, j = 0, 1, \cdots, n$. It has been shown in [2] that

$$\rho_\Lambda^2 = \sup_{F\in \mathrm{PW}} \frac{\displaystyle\int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{x^2+1/4} \prod_{\lambda\in\Lambda} \frac{x^2+(\lambda-1/2)^2}{x^2+(\lambda+3/2)^2}\, dx}{\displaystyle\int_{-\infty}^{\infty} |F(x)|^2 dx}, \tag{2}$$

where PW is the Paley-Wiener class for the upper half-plane; i.e., all f such that $\|f(x+iy)\| < M$ uniformly for all positive y. Letting

$$M(\Lambda) = \max_x \frac{1}{x^2+1/4} \prod_{\lambda\in\Lambda} \frac{x^2+(\lambda-1/2)^2}{x^2+(\lambda+3/2)^2},$$

we have $\rho_\Lambda^2 \leqq M(\Lambda)$, cf. [2].

$$M(\Lambda) = \max_x \frac{1}{x^2+1/4} \prod_{j=0}^{n} \prod_{i=0}^{n} \frac{x^2+(i+\alpha j-1/2)^2}{x^2+(i+\alpha j+3/2)^2},$$

which, by cancellation, is equal to

$$\max_x \frac{1}{x^2+1/4} \prod_{j=0}^{n} \frac{(x^2+(\alpha j-1/2)^2)(x^2+(\alpha j+1/2)^2)}{(x^2+(n+\alpha j+1/2)^2)(x^2+(n+\alpha j+3/2)^2)}. \tag{3}$$

By (1), M cannot fall off faster than c/n^4. The point $x = x(n)$ where the maximum is taken on must go to infinity with n, for, if not, (3) would yield $M = o(n^{-4})$.

Let $f(n)$ be any function which satisfies $f(n) = o(n)$. Then

$$M(\Lambda) \leqq \max_x \frac{c}{x^2} \left(\frac{x^2+\alpha^2 f^2(n)}{x^2+n^2} \right)^{2f(n)}, \quad \text{for some } c > 0.$$

By a simple differentiation, we see that the point x, at which the max is taken on, has order of magnitude $n(f(n))^{1/2}$, and, thus, $M(\Lambda) = O(1/n^2 f(n))$. Since $f(n)$ is any function which is $o(n)$, we have, in fact, $M(\Lambda) = o(1/n^2 f(n))$, from which it follows that $M(\Lambda) = O(1/n^3)$. Thus, $\rho_\Lambda = O(1/n^{3/2})$.

To obtain the lower bound, set $F(z) = 1/(z+in^{3/2})$ in (2). We get

$$\rho^2 \geqq \int_{-\infty}^{\infty} \frac{c}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2n} \frac{dx}{x^2+n^3} \Big/ \int_{-\infty}^{\infty} \frac{dx}{x^2+n^3}$$

$$\geqq \frac{n^{3/2}}{\pi} \int_{n^{3/2}}^{2n^{3/2}} \frac{c'}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2n} \frac{dx}{x^2+n^3}$$

$$\geqq \frac{c''}{n^3} \int_1^2 \frac{du}{u^2+1} = \frac{c'''}{n^3} \cdot \quad \text{Q.E.D.}$$

Although our theorem, as stated, applies only to contractions, the result is extended to approximation of functions of arbitrary modulus of continuity. This extension, which we state in the following corollary, is proved in a more general setting in [3].

COROLLARY. *Let α be irrational. Let $f(x, y)$ have modulus of continuity $\omega_f(\delta)$. Then there exists an nth degree polynomial $p(x, y)$ such that, on Q_α, $\|f-p\|_{L^2} \leqq c\omega_f(1/n^{3/2})$, c independent of f and n.*

REMARK. The above results are easily extended to curves of the form $y = p(x) + x^\alpha$, p a polynomial, α irrational, $0 \leqq x \leqq 1$.

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