METRIC DIMENSION OF COMPLETE METRIC SPACES

GLENN A. BOOKHOUT

1. Introduction and results. For integers \( n \geq 3 \), let \((X_n, \rho)\) be a metric space such that

(i) \( X_n \subseteq (K_n, \rho) \), a compact \( n \)-dimensional metric space;

(ii) \( X_n = K_n - \bigcup_{i=1}^{\infty} A_i \), where the \( A_i \)'s are mutually disjoint and closed in \( K_n \); and

(iii) \( \mu \dim(X_n, \rho) = \lfloor n/2 \rfloor \) and \( \dim X_n = n - 1 \).

(Here \( \mu \dim \) denotes metric dimension, which is defined in the next section, and \( \dim \) denotes covering dimension.) K. Sitnikov [8, p. 23] and K. Nagami and J. H. Roberts [6, p. 426] have constructed such spaces.

The result of the present paper is stated in the following theorem.

Theorem. For integers \( n \geq 3 \), let \((X_n, \rho)\) be a metric space with properties (i)–(iii) above. Then there exists a complete metric \( \sigma \) on \( X_n \) equivalent to \( \rho \) such that

\[
\mu \dim(X_n, \sigma) \leq \lfloor n/2 \rfloor + 1.
\]

K. Nagami and J. H. Roberts posed the following question. Is \( \mu \dim(X, d) = \dim X \) for all complete metric spaces \((X, d)\)? In [1, p.166] Richard E. Hodel posed an analogous question. Is \( d_2(X, d) = \dim X \) for all complete metric spaces \((X, d)\)? (The metric-dependent dimension function \( d_2 \) is defined in the next section.) It is known (see [6, Theorem 4, p. 422]) that \( d_2(X, d) \leq \mu \dim(X, d) \) for all metric spaces \((X, d)\). The present theorem gives a negative answer to these questions, since for \( n \geq 5 \),

\[
\mu \dim(X_n, \sigma) \leq \lfloor n/2 \rfloor + 1 < n - 1 = \dim X_n.
\]

M. Katětov [4, p. 166] proved that \( \dim X \leq 2 \mu \dim(X, d) \) for all nonempty metric spaces \((X, d)\). In view of this result of Katětov and the present theorem, the following problem is suggested.

Problem. For integers \( n \geq 3 \), do there exist complete metric spaces \((X_n, d)\) with \( \mu \dim(X_n, d) = \lfloor n/2 \rfloor \) and \( \dim X_n = n - 1 \)?

Received by the editors June 16, 1969.

This research is part of a doctoral dissertation prepared at Duke University under the supervision of Professor J. H. Roberts and was supported in part by the National Science Foundation under grants GP-5222 and GP-5919.
2. **Definitions.** In this paper three metric-dependent dimension functions are considered:

(i) metric dimension, denoted by \( \dim \);

(ii) \( d_2 \), introduced by K. Nagami and J. H. Roberts in [5, p. 602]; and

(iii) \( d_6 \), introduced by Richard E. Hodel in [3, p. 83].

Metric dimension, \( d_2 \), and \( d_6 \) are functions from the class of all metric spaces \( (X, d) \) into \( \{-1, 0, 1, \ldots ; \infty \} \). Condensed definitions of these functions restricted to nonempty metric spaces are as follows.

**Definition.** \( \dim(X, d) \) is the smallest integer \( n \) such that for all \( \varepsilon > 0 \) there exists an open cover \( \mathcal{U}(\varepsilon) \) of \( X \) with (1) order \( \mathcal{U}(\varepsilon) \geq n \) and (2) mesh \( \mathcal{U}(\varepsilon) < \varepsilon \).

**Definition.** \( d_2(X, d) \) is the smallest integer \( n \) such that given any \( n + 1 \) pairs \( \{ C_i, C'_i \}_{i=1}^{n+1} \) of closed sets with \( d(C_i, C'_i) > 0 \) for each \( i \), there exist closed sets \( \{ B_i \}_{i=1}^{n+1} \) such that

(i) \( B_i \) separates \( C_i \) and \( C'_i \) in \( X \) for each \( i \) and

(ii) \( \bigcap_{i=1}^{n+1} B_i = \emptyset \).

**Definition.** \( d_6(X, d) \) is the smallest integer \( n \) such that given any countable number of pairs \( \{ C_i, C'_i \}_{i=1}^{\omega} \) of closed sets with \( d(C_i, C'_i) \geq \delta \) for each \( i \) for some \( \delta > 0 \), there exist closed sets \( \{ B_i \}_{i=1}^{\omega} \) such that

(i) \( B_i \) separates \( C_i \) and \( C'_i \) in \( X \) for each \( i \) and

(ii) order \( \{ B_i \}_{i=1}^{\omega} \leq n \).

3. **Proof of the theorem.**

3.1. Reducing the problem. Fix an integer \( n \geq 3 \). Let \( (X_n, \rho) \) be a metric space with properties (i)–(iii) above. We may assume that every \( A_i \) is nonempty. Define

\[
 f_i(x) = \frac{1}{\rho(x, A_i)}, \quad (x \in K_n - A_i, i \geq 1);
\]

\[
 \alpha_i(x, y) = 2^{-i} \frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}, \quad (x, y \in K_n - A_i, i \geq 1);
\]

\[
 \sigma(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} \alpha_i(x, y), \quad (x, y \in X_n).
\]

It is known (see [2, Theorem 2-76, p. 85]) that \( \sigma \) is a complete metric on \( X_n \) equivalent to \( \rho \).

We shall prove that \( \dim(X_n, \sigma) \leq \lceil n/2 \rceil + 1 \). It is proved in [3, p. 85] that \( d_6(X, d) = \dim(X, d) \) for all separable metric spaces \( (X, d) \). Now \( X_n \) is separable, so it suffices to prove that \( d_6(X_n, \sigma) \leq \lceil n/2 \rceil + 1 \). Let \( \{ C_i, C'_i \}_{i=1}^{\omega} \) be a countable number of pairs of closed
sets in $X_n$ with $\sigma(C_i, C'_i) \geq \epsilon$ for each $i$ for some $\epsilon > 0$. We want to show that there exist closed sets \( \{B_i\}_{i=1}^n \) in $X_n$ such that

(i) $B_i$ separates $C_i$ and $C'_i$ in $X_n$ for each $i$ and

(ii) order $\{B_i\}_{i=1}^n \leq \lceil n/2 \rceil + 1$.

Since $\sum_{i=1}^n \alpha_i$ converges uniformly in $X_n$, there exists an integer $N > 1$ such that $\sum_{i=N+1}^\infty \alpha_i(x, y) < \epsilon/2$ for all $x, y \in X_n$. Define

$$\sigma^N(x, y) = \rho(x, y) + \sum_{i=1}^N \alpha_i(x, y), \quad (x, y \in X_n).$$

$$A = \bigcup_{i=1}^N A_i.$$

Then clearly $\sigma^N$ is a metric on $X_n$ equivalent to $\rho$. Also, since $\sigma(C_i, C'_i) \geq \epsilon$ for all $i$, it follows that

$$\sigma^N(C_i, C'_i) \geq \frac{\epsilon}{2} \text{ for all } i.$$

### 3.2. Definitions

Define

$$\delta = \min\{\rho(A_i, A_j) : i, j \in \{1, 2, \ldots, N\}, i \neq j\},$$

$$\gamma = \min \left\{ \frac{\delta}{4}, \frac{\epsilon}{6}, \frac{\epsilon \delta^2}{24(N-1)} \right\}.$$

### 3.3. Assertion 1

For all numbers $a$ such that $0 < a \leq \delta/4$, there exists an $\epsilon(a) > 0$ such that $\rho(C_i, C'_i) < \gamma$ in $S(\epsilon(a))$ (\( \equiv \{x \in K_n : a - \epsilon(a) < \rho(x, A_i) < a + \epsilon(a)\} \)) for $i \geq 1$.

**Proof.** Fix $a$ such that $0 < a \leq \delta/4$. Choose $\epsilon(a) > 0$ such that $\epsilon(a) < \min\{a/2, \epsilon a^2/48\}$. Suppose there exists an integer $i \geq 1$ such that $\rho(C_i, C'_i) < \gamma$ in $S(\epsilon(a))$. Then there exist points $x \in C_i$ and $y \in C'_i$ such that $\{x, y\} \subset S(\epsilon(a))$ and $\rho(x, y) < \gamma$. From the definition of $\gamma$ and the choice of $\epsilon(a)$, it follows that $\rho(x, A_i) < \delta/4$, $\rho(x, A) < 3\delta/8$, and $\rho(y, A) < 3\delta/8$. Therefore by the definition of $\delta$, there exists an integer $k \in \{1, 2, \ldots, N\}$ such that $\rho(x, A_k) < 3\delta/8$ and $\rho(y, A_k) < 3\delta/8$. Thus for $i \in \{1, 2, \ldots, N\}$ and $i \neq k$, $\rho(x, A_i) > \delta/2$ and $\rho(y, A_i) > \delta/2$. It follows that $a - \epsilon(a) < \rho(x, A_k) < a + \epsilon(a)$ and $a - \epsilon(a) < \rho(y, A_k) < a + \epsilon(a)$. Hence $|\rho(x, A_k) - \rho(y, A_k)| < 2\epsilon(a)$. Finally, $\rho(x, A_k) > a/2$ and $\rho(y, A_k) > a/2$. From the definitions of $\sigma^N$ and $\gamma$ and the inequalities above, it follows that
\[ \sigma^N(x, y) \leq \rho(x, y) + \sum_{i=1}^{N} |f_i(x) - f_i(y)| \]
\[ \leq \rho(x, y) + \sum_{i=1}^{N} \left| \frac{\rho(x, A_i) - \rho(y, A_i)}{\rho(x, A_i) \cdot \rho(y, A_i)} \right| \]
\[ \leq \rho(x, y) + \sum_{i=1}^{N} \left( \frac{\rho(x, y)}{\rho(x, A_i) \cdot \rho(y, A_i)} + \frac{\left| \rho(x, A_k) - \rho(y, A_k) \right|}{\rho(x, A_k) \cdot \rho(y, A_k)} \right) \]
\[ < \gamma + \frac{(N-1)\gamma}{\delta^2/4} + \frac{2\epsilon(a)}{a^2/4} \]
\[ < \epsilon/6 + \epsilon/6 + \epsilon/6 = \epsilon/2, \]

contradicting (1).

3.4. Construction of \( C_{ij}, C'_{ij} \). Now (i) \( \{S(\epsilon(a)) : 0 < a \leq \delta/4\} \) is a collection of open sets in \( K_\mathbb{N} \) covering \( \{x \in K_\mathbb{N} : 0 < \rho(x, A) \leq \delta/4\} \) and (ii) \( \{x \in K_\mathbb{N} : \delta/(4 \cdot 2^j) \leq \rho(x, A) \leq \delta/(4 \cdot 2^{j-1})\} \) is compact for \( j \geq 1 \). Using (i) and (ii), it is easy to prove that there exist a sequence \( \{a_j\}_{j=1}^{\infty} \) of positive numbers \( \leq \delta/4 \) such that
- (a) \( \bigcup_{j=1}^{\infty} S(\epsilon(a_j)) \) covers \( \{x \in K_\mathbb{N} : 0 < \rho(x, A) \leq \delta/4\} \) and
- (b) the sequence \( \{a_j\}_{j=1}^{\infty} \) converges to 0.

We can choose a sequence \( \{\delta_j\}_{j=1}^{\infty} \) of distinct positive numbers such that \( \delta_1 = \delta/4 \), \( \{\delta_j\}_{j=1}^{\infty} \) is a strictly decreasing sequence converging to 0, and for each \( j \geq 2 \) there exists an integer \( k \geq 1 \) such that
\[ \delta_{j-1} - \epsilon(a_k) < \delta_{j+1} < \delta_{j-1} < a_k + \epsilon(a_k). \]

Now we define distinct positive numbers \( \{\delta_{ij}\}_{i,j=1}^{\infty} \) as follows. Fix \( j \geq 1 \). Define \( \delta_{ij} = \delta_j \). For \( i > 1 \) choose the \( \delta_{ij} \)'s to be distinct numbers strictly between \( \delta_j \) and \( \delta_{j+1} \).

Now define
\[ E_{i1} = \{x \in X_n : \rho(x, A) \geq \delta_{i1}\}, \quad (i \geq 1) ; \]
\[ E_{ij} = \{x \in X_n : \delta_{ij} \leq \rho(x, A) \leq \delta_{i-1,j-1}\}, \quad (i \geq 1, j > 1) ; \]
\[ C_{ij} = C_i \cap E_{ij}, \quad C'_{ij} = C'_i \cap E_{ij}, \quad (i, j \geq 1) . \]

3.5. Assertion 2. There exists a \( \tau > 0 \) such that \( \rho(C_{ij}, C'_{ij}) \geq \tau \) for \( i, j \geq 1 \).

**Proof.** Define \( \tau = \min \{\gamma, \epsilon \delta^2/4N\} \).

**Case 1.** \( j = 1 \). Suppose there exists an integer \( i \geq 1 \) such that
\( \rho(C, C') < \tau \). Let \( x \in C \) and \( y \in C' \) be such that \( \rho(x, y) < \tau \). Note that \( \rho(x, A) > \delta_2 \) and \( \rho(y, A) > \delta_2 \), since \( \{x, y\} \subset E_i \). Hence

\[
\sigma^N(x, y) \leq \rho(x, y) + \sum_{i=1}^N \frac{|\rho(x, A_i) - \rho(y, A_i)|}{\rho(x, A_i) \cdot \rho(y, A_i)} \\
\leq \rho(x, y) + \sum_{i=1}^N \frac{\rho(x, y)}{\rho(x, A_i) \cdot \rho(y, A_i)} \\
< \tau + N\tau / \delta_2^2 \\
< \epsilon / 4 + \epsilon / 4 = \epsilon / 2,
\]
a contradiction to (1).

**Case 2.** \( j > 1 \). Fix \( i \geq 1 \) and \( j > 1 \). Now by the definition of \( E_{ij} \) and by (2),

\[
E_{ij} \subset \{ x \in X_n : \delta_{j+1} \leq \rho(x, A) \leq \delta_{j-1} \} \\
\subset S(\epsilon(a))
\]
for some \( a \) such that \( 0 < a \leq \delta / 4 \). Therefore by the definitions of \( C_{ij} \) and \( C'_j \) and Assertion 1, \( \rho(C_{ij}, C'_j) \geq \gamma \geq \tau \).

**3.6. Lemma [7].** Let \( X \) be a topological space, let \( C \) and \( C' \) be disjoint closed sets in \( X \), and let \( \{D_j\}_{j=0}^\infty \) be an open cover of \( X \) such that \( D_0 = \emptyset \) and \( \overline{D}_j \subset D_{j+1} \) for all \( j \geq 1 \). Suppose there exist closed sets \( \{B_j\}_{j=1}^\infty \) in \( X \) such that \( B_j \subset \overline{D}_j - D_{j-1} \) for \( j \geq 1 \) and \( B_j \) separates \( C \cap (D_j - D_{j-1}) \) and \( C' \cap (D_j - D_{j-1}) \) in \( D_j - D_{j-1} \) for \( j \geq 1 \). Then there exists a closed set \( B \) in \( X \) such that \( B \) separates \( C \) and \( C' \) in \( X \) and \( B \subset \bigcup_{j=1}^\infty (B_j \cup (\overline{D}_j - D_j)) \).

**3.7. Conclusion of the proof of the theorem.** By Assertion 2 and the equality \( d_\delta(X_n, \rho) = [n/2] \), there exist closed sets \( \{B'_{ij}\}_{i,j=1}^\infty \) in \( X_n \) such that \( B'_{ij} \) separates \( C_{ij} \) and \( C'_{ij} \) in \( X_n \) for \( i, j \geq 1 \) and order \( \{B'_{ij}\}_{i,j=1}^\infty \leq [n/2] \). For \( i \geq 1 \) define \( D_{i0} = \emptyset \). For \( i, j \geq 1 \) define \( D_{ij} = \{ x \in X_n : \rho(x, A) > \delta_{ij} \} \) and \( B_{ij} = B'_{ij} \cap (\overline{D}_{ij} - D_{i,j-1}) \), where for every \( i \) and \( j \) the closure of \( D_{ij} \) is taken with respect to \( X_n \). Then clearly \( B_{ij} \) separates \( C_i \) and \( C'_j \) in \( D_{ij} - D_{i,j-1} \) for \( i, j \geq 1 \) and

\[
\text{order}\{B_{ij}\}_{i,j=1}^\infty \leq [n/2].
\]

Now fix \( i \geq 1 \). Clearly \( X_n, C_i, C'_i, \{D_{ij}\}_{j=0}^\infty \), and \( \{B_{ij}\}_{j=1}^\infty \) satisfy the conditions of the lemma. Therefore there exists a closed set \( B_i \) in \( X_n \) such that \( B_i \) separates \( C_i \) and \( C'_i \) in \( X_n \) and

\[
B_i \subset \bigcup_{j=1}^\infty (B_{ij} \cup (\overline{D}_{ij} - D_{ij})).
\]
But for $j \geq 1$,

$$\overline{D}_{ij} - D_{ij} \subset \{ x \in X_n : \rho(x, A) = \delta_{ij} \}. $$

Hence

$$B_i \subset \bigcup_{j=1}^{\infty} (B_{ij} \cup \{ x \in X_n : \rho(x, A) = \delta_{ij} \}).$$

Therefore, by (3) and the fact that the $\delta_{ij}$'s are distinct for $i,j \geq 1$, we have that order $\{ B_i \}_{i=1}^{\infty} \leq \lfloor n/2 \rfloor + 1$, and the proof is complete.

REFERENCES


Duke University