A FIVE SPHERE DECOMPOSITION OF $E^{2n-1}$

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I. Introduction. R. H. Bing and M. L. Curtis have exhibited a decomposition of Euclidean 3-dimensional space $E^3$ into twelve mutually disjoint circles and points not on the circles such that the associated decomposition space cannot be embedded in $E^4$ [1]. Their method consists in showing that the space contains a certain 2-dimensional polyhedron that Flores has proved to be impossible to embed in $E^4$ [2]. The construction of Bing and Curtis was later modified by R. H. Rosen, who, by improving the result of Flores, also exhibited a decomposition of $E^3$ that cannot be embedded in $E^4$, and in which he used only six circles instead of twelve [4]. In the opposite direction, R. P. Goblirsch showed that every decomposition using only three circles as nondegenerate elements can be embedded in $E^4$ [3]. Thus, for the numbers four and five the question remained open. Rosen conjectured in [4] that one could build an example by using five circles in $E^3$ such that each circle links exactly two others. In this paper we show this conjecture to be correct. Moreover, our argument begins in a lower dimension: We construct an analogous decomposition of $S^1$ with five nontrivial elements such that the associated decomposition space cannot be embedded in $S^2$. The example conjectured by Rosen then becomes the second step in an induction argument. Thus we show that for each integer $n, n \geq 1$, there exists a decomposition of $S^{2n-1}$ with nondegenerate elements consisting of five $(n-1)$-spheres such that the associated decomposition space cannot be embedded in $S^{2n}$. This inductive viewpoint was inspired by a paper of Joseph Zaks [5], in which decompositions of $E^{2n-1}$ with finitely many nondegenerate elements were constructed for all $n \geq 1$.

II. Embedding an $n$-complex in $S^{2n-1}$. Let $N^1$ denote the 1-skeleton of a 4-simplex with vertices $a_1, b_1, c_1, d_1$, and $e_1$. Let $N^2$ denote the join $V(N^1, \{a_2, b_2, c_2\})$ of $N^1$ with the three point space $\{a_2, b_2, c_2\}$. Proceeding inductively, $N^n$ is defined as $V(N^{n-1}, \{a_n, b_n, c_n\})$. It is shown in [2] and [4] that $N^n$ cannot be embedded in $E^{2n}$. We name five $n$-simplices of $N^n$:

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Setting $N^n = N^n - \sum_1^5 \text{Int } D_i$, we find that $N^n$ embeds in $S^{2n}$. In fact, it embeds in $S^{2n-1}$! Rather than prove this fact, which would require cumbersome notation, we establish a weaker result, which suffices for our purposes. We call two points of a geometric complex distant if they lie in disjoint, closed simplexes of the complex.

**Lemma.** For $n \geq 1$, there exists a map $f_n : N^n - S^{2n-1}$ such that no two distant points of $N^n$ have the same image.

![Figure 1](image-url)

**Figure 1**

**Proof.** An induction argument begins with the fact that $N^1$ is homeomorphic to $S^1$ as is shown in Figure 1; call such a homeomorphism $f_1$. For $n = 2$, the reader is advised first to familiarize himself
with the visualizations given in [1]. In fact, for \( n = 2 \), Bing and Curtis construct geometrically just what we will do notationally, except that their complex "lacks" three 2-cells instead of the five 2-cells that \( N^2 \) "lacks." We regard \( S^3 \) as the join \( V(S^1, S^1) \), with \( f_1 \) viewed as an embedding of \( N^2 \) into the first factor of \( V(S^1, S^1) \), and with \( \{a_2, b_2, c_2\} \) viewed as a subset of the second factor. Then \( V(f_1(N^2), \{a_2, b_2, c_2\}) \) is a subset of \( V(S^1, S^1) \) in a natural way; this provides us with an embedding \( f_2 \) of all but ten 2-simplices of \( N^2 \) into \( S^3 \). We select points \( p, q, \) and \( r \) in the second factor of \( V(S^1, S^1) \) so that this factor is composed of the six arcs \( a_2p, pb_2, b_2q, qc_2, c_2r, ra_2 \). We define \( f_2(a_1c_1) \) as \( V(f_1(Bd a_1c_1), p) \), \( f_2(a_1c_1a_2) \) as \( V(f_1(Bd a_1c_1), a_2p) \) as illustrated in

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Figure 2, and \( f_2(a_1c_1b_2) \) as \( V(f_1(\text{Bd } a_1c_1), pb_2) \). Next, we define \( f_2(a_1d_1) \) as \( V(f_1(\text{Bd } a_1d_1), p) \), \( f_2(a_1d_1a_2) \) as \( V(f_1(\text{Bd } a_1d_1), a_2p) \) as illustrated in Figure 2, and \( f_2(a_1d_1b_2) \) as \( V(f_1(\text{Bd } a_1d_1), pb_2) \). Similarly, we define \( f_2(b_1d_1) \) as \( V(f_1(\text{Bd } b_1d_1), q) \) and insert \( f_2(b_1d_1b_2) \) and \( f_2(b_1d_1c_2) \), \( f_2(b_1e_1) \) as \( V(f_1(\text{Bd } b_1e_1), q) \) and insert \( f_2(b_1e_1b_2) \) and \( f_2(b_1e_1c_2) \). Finally, we define \( f_2(c_1e_1) \) as \( V(f_1(\text{Bd } c_1e_1), r) \), then insert \( f_2(c_1e_1c_2) \) and \( f_2(c_1e_1a_2) \). Thus \( f_2 \) has been defined, and one may verify that it satisfies the lemma; in fact, a small adjustment would make \( f_2 \) an embedding.

For \( n = 3 \), we let \( f_2 \) map into the first factor of \( V(S^3, S^1) \), and \( a_3, p', b_3, q', c_3, r' \) be consecutive points in the second factor. Then \( f_3(a_1c_1c_2) \) is defined as \( V(f_2(\text{Bd } a_1c_1c_2), p') \); then \( f_3(a_1c_1c_2a_3) \) and \( f_3(a_1c_1c_2b_3) \) are inserted as before. The continuation is just a notational exercise.

III. Insertion of five annuli.

Theorem. For each integer \( n, n \geq 1 \), there exists a decomposition of \( S^{2n-1} \) with nondegenerate elements consisting of five \((n-1)\)-spheres such that the associated decomposition space cannot be embedded in \( S^{2n} \).

Proof. Let \( A' \) denote the subarc of \( S^1 \) with interior point \( f_1(a_1) \)
and end points \( f_i(a_i) + f_i(b_i) \); similarly \( B' \) has interior point \( f_i(b_i) \) and end points \( f_i(a_i) + f_i(c_i) \); analogously we define \( C', D', \) and \( E' \). We set 
\[
A = V(A', S^{2n-3}) \subset S^{2n-1},
\]
and similarly for \( B, C, D, \) and \( E \). The map 
\[
f_n: N^n \to S^{2n-1}
\]
can be extended to \( N^n \) so that 
\[
f_n(\text{Int} \, D_1) \subset \text{Int} \, B, \quad f_n(\text{Int} \, D_2) \subset \text{Int} \, E, \quad f_n(\text{Int} \, D_3) \subset \text{Int} \, C, \quad f_n(\text{Int} \, D_4) \subset \text{Int} \, A, \quad f_n(\text{Int} \, D_5) \subset D,
\]
with \( f_n/\text{Int} \, D_i \) an embedding for all \( i \). We discard an open disk \( \theta_i \) from \( f_n(\text{Int} \, D_i) \), leaving an annulus \( U_i \) with boundary consisting of \( \alpha_i = \text{Bd} \, f_n(D_i) \) plus another \( n \)-sphere which we call \( \beta_i \). By choosing \( \theta_i \) sufficiently large, we may ensure that 
\[
U_1 \cdot U_3 = U_1 \cdot U_4 = U_2 \cdot U_4 = U_2 \cdot U_5 = U_3 \cdot U_5 = \emptyset,
\]
as the corresponding \( \alpha_i \)'s are disjoint. In fact, for all other pairs \( U_i \cdot U_j \) with \( i \neq j \), this intersection will be precisely \( \alpha_i \cdot \alpha_j \). For example, to see that \( U_1 \cdot U_2 = \alpha_1 \cdot \alpha_2 \), observe that \( U_1 - \alpha_1 \subset \text{Int} \, B, \quad U_2 - \alpha_2 \subset \text{Int} \, E, \) and \( \text{Int} \, B \cdot \text{Int} \, E = \emptyset \).

We wish to show that the decomposition of \( S^{2n-1} \) with nondegenerate elements \( \beta_1, \beta_2, \ldots, \beta_5 \) does not embed in \( S^{2n} \). We show that this would imply a map of \( N^n \) into \( S^{2n} \) such that no two distant points of \( N^n \) have the same image, contradicting [4]. All that needs to be checked is how the annuli \( U_i - \alpha_i \) intersect \( N^n \) in \( S^{2n-1} \). We already know that they do not intersect each other. Furthermore, it is easy to require that \( U_i - \alpha_i \) intersects a simplex \( \Delta \) of \( N^n \) only if they share a common vertex, by increasing the size of \( \theta_i \) if necessary. It remains to show that if \( \beta_1 \cdot \Delta \neq \emptyset \) and \( \beta_1 \cdot \Delta_2 \neq \emptyset \), then \( \Delta_1 \) and \( \Delta_2 \) have a common vertex. For notational convenience, assume that \( i = 1 \), so \( \beta_1 \subset \text{Int} \, B \). By general position, we may assume that \( \Delta_1 \) and \( \Delta_2 \) are both \( n \)-simplices on \( N^n \). But any two \( n \)-simplices in \( \text{Int} \, B \) have a common vertex.

IV. Questions. Let us first observe that our result is the best possible for \( n = 1 \); any decomposition of \( S^1 \) with four (or less) nondegenerate elements can be embedded in \( S^2 \) without great difficulty. For \( n \geq 2 \), however, unsolved problems abound. For example, by using methods of Goblirsch [3], one can embed all four circle decompositions of \( S^3 \) in \( S^4 \) with one exception, illustrated in Figure 3. Can this example also be embedded in \( S^4 \)? Note that care must be taken in this example that the four circles do not lie on a common torus in \( S^3 \); that is, these four circles do not all link each other in the most natural way. Indeed, if they did, the technique of [3] would give an embedding.

If we do not require circles but merely simple closed curves, then
Figure 4 gives a decomposition of $S^3$ with only three nondegenerate sets. Can this example be embedded in $S^4$? Note that Goblirsch’s technique can not be applied to this example. Indeed, this question is unsolved if we do not require simple closed curves, but merely continuua.

If $K$ is an $n$-complex which locally embeds in $S^{2n-1}$, does $K$ embed in $S^{2n}$?
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References


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