THE DEFICIENCY OF METACYCLIC GROUPS

J. W. WAMSLEY

If a finite group G is generated by n elements and defined by m relations between them then G has a presentation

$$G = \{x_1, \cdots, x_n \mid R_1, \cdots, R_m\}.$$

Clearly $m \geqq n$ and the value $n-m$ is said to be the deficiency of the given presentation. The deficiency of G, denoted $\operatorname{def}(G)$, is the maximum of the deficiencies of all the finite presentations of G.

It is implicit in I. Schur [1] that the minimal number of generators of the Schur "multiplicator", as an abelian group, is less than or equal to $-\operatorname{def}(G)$. B. H. Neumann [2] asks whether a finite group with trivial multiplicator has deficiency zero, R. G. Swan [3] answers this question by giving a family of finite groups with trivial multiplicator and negative deficiency.

In this paper we confirm Neumann's conjective that $H_2(G, Z)=0$ if and only if $\operatorname{def}(G)=0$ in the metacyclic case by proving

THEOREM. *Let G be a finite group with presentation*

$$G = \{a, b \mid b^{-1}aba^{-1-\alpha}, b^{\gamma}a^{-\delta}, a^{\beta}\}$$

with α, β, γ, δ nonnegative and such that the order of G is $\gamma\beta$, then the following are equivalent:

(i) *Deficiency of G is zero.*

(ii) $H_2(G, Z)=0$.

(iii) $\text{G.C.D}(\alpha, \beta, \delta, \alpha\delta/\beta, \{(1+\alpha)^{\gamma}-1\}/\beta, \{(1+\alpha)^{\gamma}-1\}/\alpha)=1$.

LEMMA 1. *Let G be a finite group with presentation*

$$G = \{a, b \mid b^{-1}aba^{-\alpha}, b^{\gamma}a^{-\delta}, a^{\beta}\}$$

with α, β, γ, δ nonnegative integers such that G is of order $\gamma\beta$ then $H_2(G, Z)=Z_n$, the cyclic group of order n, where

$$n = \text{G.C.D}(\alpha - 1, \beta, \delta, (\alpha^{\gamma} - 1)/\beta, (\alpha^{\gamma} - 1)/(\alpha - 1), \delta(\alpha - 1)/\beta).$$

PROOF. A resolution given by C. T. C. Wall [4] is

$$0 \leftarrow Z \xleftarrow{\epsilon} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3}$$

Received by the editors September 16, 1968 and, in revised form, May 13, 1969.

where A_n is free on $a_{n,0}, a_{n-1,1}, \cdots, a_{0,n}$; with d given in matrix form as

$$d_1 = (a-1 \quad b-1),$$

$$d_2 = \begin{pmatrix} N & 1-bL_1 & M \\ 0 & a-1 & \sum_{j=0}^{\gamma-1} b^j \end{pmatrix},$$

$$d_3 = \begin{pmatrix} a-1 & bL_1-1 & -(1/\beta)(\alpha^\gamma-1) & -b(\delta_\alpha-\delta/\beta) \\ 0 & N & -\sum_{j=0}^{\gamma-1} b^jL_j & M \\ 0 & 0 & a-1 & b-1 \end{pmatrix}$$

where

$$N = \sum_{i=0}^{\beta-1} a^i, \qquad L_j = \sum a^i : 0 \leqq i < \alpha^j$$

and

$$\begin{aligned} M &= -1-a-\cdots-a^{\delta-1}, & \delta &\neq 0, \\ &= 0, & \delta &= 0, \end{aligned}$$

whence the result follows.

LEMMA 2. *Let G be a finite group with presentation*

$$G = \{a, b \mid b^{-1}ab = a^{1+\alpha},\ b^\gamma = a^\delta,\ a^\beta = 1\}$$

where α, β, γ, δ are nonnegative, G of order $\gamma\beta$, and $H_2(G, Z)=0$, then $\operatorname{def}(G)=0$.

PROOF. If p and q are integers such that p^r divides q but p^{r+1} does not divide q we write $p^r \| q$.

(i) If $p^r\|\beta$, $p^{r+s}\|\delta$, $s>0$, δ nonzero, p prime then replace δ with $\delta(p^s+\beta/p^r)/p^s=\delta'$ giving $p^r\|\delta'$. Suppose q is such that $(q, p)=1$ and $q\mid\beta$, $q^n\|\delta$ then $q^n\|\delta'$ i.e. we may choose δ to be zero or such that if p is a prime with $p^r\|\beta$ then $p^{r+1}\nmid\delta$.

(ii) If $\beta=p_1^{\beta_1}\cdots p_n^{\beta_n}$, p_i distinct primes, $\beta_i>0$. Then we may take $\delta=p_1^{\delta_1}\cdots p_n^{\delta_n}$ or 0, $\delta_i\geqq 0$, $\delta_i\leqq\beta_i$ for if $\delta=kt$, $(k, \beta)=1$ replace a with a^k in the generating set.

(iii) Similarly we may take

$$\alpha = p_1^{\alpha_1}\cdots p_n^{\alpha_n}h, \qquad (h, \beta) = 1, \qquad \alpha_i \leqq \beta_i.$$

(iv) If $(\alpha, \beta)=1$, then $\delta=0$ since $\beta\mid\alpha\delta$. We then solve $m\alpha\equiv 1 \pmod{\beta}$ for m. Let $G_1=\{a, b\mid b^{-1}a^mb=a^{m+1},\ b^\gamma=a^\beta\}$ giving $b^{-1}ab=a^{1+\alpha}$, $a^\beta=1$, $b^\gamma=1$ i.e. $G_1\cong G$, so we may assume $(\alpha, \beta)\neq 1$.

(v) If $\delta=0$, then $(ab)^\gamma=a^{\{(1+\alpha)^\gamma-1\}/\alpha}$, suppose $p^r|\beta$, $p^s|\alpha$, $s\leqq r$, $s\neq 0$, $p^s\neq 2$, then $p\nmid\{(1+\alpha)^\gamma-1\}/\beta$, otherwise $H_2(G, Z)\neq 0$, i.e, $p^{r-s}\|\gamma$, and $p^r\nmid\{(1+\alpha)^\gamma-1\}/\alpha$, giving $(ab)^\gamma\neq 1$. In the case $p^s=2$ we have

$$2\nmid\{(1+\alpha)^\gamma-1\}/\beta, \quad \text{otherwise } H_2(G, Z)\neq 0,$$

i.e. $2^r\|(1+\alpha)^\gamma-1$, giving $2^r\nmid\{(1+\alpha)^\gamma-1\}/\alpha$ or $(ab)^\gamma\neq 1$. Replacing b with ab in the generating set gives $\delta\neq 0$, so we may assume $\delta\neq 0$.

(vi) Let

$$\beta = p_1^{\beta_1}\cdots p_n^{\beta_n}, \qquad p_i \text{ distinct primes}, \beta_i>0,$$

$$\alpha = kp_1^{\alpha_1}\cdots p_n^{\alpha_n}, \qquad \alpha_i\leqq\beta_i, \quad (k,\beta)=1,$$

$$\delta = p_1^{\delta_1}\cdots p_n^{\delta_n}, \qquad \delta_i\leqq\beta_i,$$

$$\gamma = hp_1^{\gamma_1}\cdots p_n^{\gamma_n}, \qquad (h,\beta)=1.$$

Let m be such that $m\alpha k\equiv\alpha \pmod{\alpha^2\delta/k}$, i.e. $mk\equiv 1 \pmod{\alpha\delta/k}$. Let $G_2=\{a, b\mid b^{-1}a^m b=a^{m+\alpha/k}, b^\gamma=a^\delta\}$. Then we have

(a) a has order dividing $\alpha\delta/k$.

(b) $b^{-1}ab=a^{1+\alpha}$.

(c) a has order dividing $(1+\alpha)^\gamma-1$.

(d) $\beta\mid\alpha\delta/k$.

(e) If $p\mid\alpha\delta/k$ then $p\mid\beta$.

Suppose $p^r\|\beta$, $r>0$ and $p^s\|\alpha$, $p^t\|\delta$, $p^m\|\gamma$. If $p^{r+1}|\alpha\delta$, then $p^r\|(\alpha\delta/k, (1+\alpha)^r-1)$. Suppose $p^{r+1}|\alpha\delta$, then $s+t>r$, giving $s>0$, $t>0$, since $s\leqq r$, $t\leqq r$, whence $p\mid\alpha$, $p\mid\beta$, $p\mid\delta$, $p\mid\delta\alpha/\beta$. However since $H_2(G, Z)=0$, $p\nmid\{(1+\alpha)^\gamma-1\}/\beta$ giving $p^r\|(1+\alpha)^\gamma-1$ or $p^r\|(\alpha\delta/k, (1+\alpha)^\gamma-1)$ whereby it follows $\beta=(\alpha\delta/k, (1+\alpha)^\gamma-1)$ giving $G_2\cong G$.

The proof of the theorem follows from the lemmas.

Finally I wish to thank Dr. I. D. Macdonald, who suggested the problem, for his assistance and the referee for his helpful suggestions.

References

1. I. Schur, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionem*, J. Reine Angew. Math. **132** (1907), 85–137.

2. B. H. Neumann, *On some finite groups with trivial multiplicator*, Publ. Math. Debrecen **4** (1956), 190–194. MR **18**, 12.

3. R. G. Swan, *Minimal resolutions for finite groups*, Topology **4** (1965), 193–208. MR **31** #3482.

4. C. T. C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. **57** (1961), 251–255. MR **31** #2304.

Flinders University of South Australia, Bedford Park, South Australia