OSCILLATION CRITERIA FOR NONLINEAR MATRIX DIFFERENTIAL INEQUALITIES

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Abstract. Oscillation criteria are established for nonlinear matrix differential equations of the form $[A(x)V']' + B(x, V, V')V = 0$ and associated differential inequalities. The hypothesis used recently by E. C. Tomastik, that $A$ and $B$ are positive definite, is weakened to the following: $A$ is positive semidefinite.

Oscillation criteria for the matrix differential equation

$(1) \quad LV = [A(x)V']' + B(x, V, V')V = 0,$

and more generally for the inequality $V^T LV \leq 0$ (as a form), will be derived by a technique different from that given recently by Tomastik [3]. It will be assumed that $A$, $B$, and $V$ are $m \times m$ matrix functions, $A(x)$ is symmetric, positive semidefinite, and continuous on an interval $[a, \infty)$, and $B(x, V, V')$ is symmetric and continuous for $x$ on $[a, \infty)$ and for all values of the entries of $V$ and $V'$. Although Tomastik requires $B(x, V, V')$ to be positive definite on $a \leq x < \infty$ for every matrix $V$ with $\det V \neq 0$, we require only that $B(x, V, V')$ satisfies condition (2) below. As already noted, we have weakened the positive definiteness of $A(x)$ to positive semidefiniteness. The technique used here has the advantage that it can be adapted [2], [1, Chapter 5] to partial differential inequalities of elliptic or parabolic type. As in [3] it is assumed that every solution of (1) can be continued to $x = \infty$.

Theorem 1. The inequality $V^T LV \leq 0$ is oscillatory if $A(x)$ is bounded above and there exists a diagonal element $B_{ii}$ of $B$ such that

$(2) \quad \int_a^\infty B_{ii}[x, V(x), V'(x)]dx = + \infty$

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for every differentiable matrix $V(x)$ with $\det V(x) \neq 0$ for all sufficiently large $x$.

**Proof.** Suppose to the contrary that $V^T L V \leq 0$ is not oscillatory, i.e. [3] there exists a number $b \geq a$ and a prepared matrix $V(x)$ satisfying $V^T L V \leq 0$ such that $\det V(x) \neq 0$ in $(b, \infty)$. Then a unique solution $w(x)$ of $u(x) = V(x)w(x)$ exists in $(b, \infty)$ for any $m$-vector $u(x)$. The following identity is easily verified by differentiation for any piecewise $C^1$ vector function $u$:

$$(Vw')^T AVw' + [(Vw)^T AV'w]' = u^T A' u - u^T B u + u^T L w + w^T (V^T AV' - V^T' AV)w'.$$

The last term is identically zero since $V$ is prepared [3]. Since $V^T L V \leq 0$, it follows that

$$(3) \quad F[u, V] = \int_b^c [u^T A(x)u' - u^T B(x, V, V')u] dx \geq 0$$

if $u$ is any piecewise $C^1$ vector function satisfying $u(b) = u(c) = 0$ $(b < c)$. In particular, choose $u$ to be $u_i$, where

$$u_i(x) = \begin{cases} 0 & \text{if } a \leq x \leq b \\ e_i(x - b) & \text{if } b < x \leq b + 1 \\ e_i & \text{if } b + 1 < x \leq c - 1 \\ e_i(c - x) & \text{if } c - 1 < x \leq c \\ \end{cases}$$

$(c > b + 2)$ and $e_i$ is the unit vector with 1 in the $i$th position and 0 elsewhere. Then

$$F(u_i, V) \leq 2\alpha - \frac{\beta}{3} - \int_{b+1}^{c-1} B_{ii}(x, V, V') dx$$

$$- \int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx$$

where $\alpha$ is an upper bound for $A(x)$ on $[b, \infty)$ and $\beta$ is a lower bound for $B(x, V(x), V'(x))$ on $[b, b+1]$. In view of the hypothesis (2), there exists a number $c_0$ such that

$$F(u_i, V) \leq - \int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx$$

for all $c \geq c_0$. Define
\[ g(w, x) = \int_{w}^{x} B_{ii}(t, V(t), V'(t))dt, \quad w \leq x. \]

We assert there exists a number \( c \geq c_0 \) such that \( g(c - 1, x) > 0 \) for all \( x > c - 1 \). In fact, \( g(c_0 - 1, x) \) has a largest zero \( x = c - 1 \) by (2) \( (c \geq c_0) \) and \( g(c_0 - 1, x) = g(c - 1, x) > 0 \) for \( x > c - 1 \). An easy integration by parts shows that

\[ \int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2dx = 2 \int_{c-1}^{c} (c - x)g(c - 1, x)dx > 0. \]

Hence \( F[u, V] < 0 \), contradicting (3).

Theorem 1 remains true if (2) is replaced by the obviously stronger condition

\[ \int_{a}^{\infty} \text{tr} B[x, V(x), V'(x)]dx = + \infty, \]

where \( \text{tr} B \) denotes the trace of \( B \). In the case that \( B \) is positive definite, Theorem 1 also remains true if (2) is replaced by the equivalent condition

\[ \int_{a}^{\infty} \lambda[x, V(x), V'(x)]dx = + \infty \]

where \( \lambda \) denotes the largest eigenvalue of \( B \). In general, it is clear that (4) can be replaced by (5) provided also \( \int_{a}^{\infty} \lambda_i(x)dx \) is finite for every negative eigenvalue \( \lambda_i \) of \( B[x, V(x), V'(x)] \).

If \( A \) is the identity matrix and \( B \) is positive definite, the hypothesis (5) is needed only for matrices \( V(x) \) such that the smallest eigenvalue of \( V^T(x)V(x) \) is bounded away from zero for large \( x \), as shown by Tomastik [3].

**Corollary.** The equation \( LV = 0 \) is oscillatory if \( A \) and \( B \) are positive definite, \( A \) is bounded above, and (5) holds for every differentiable matrix \( V \) with \( \det V(x) \neq 0 \) for all sufficiently large \( x \).

This is an obvious special case of Theorem 1; it is also a specialization of Tomastik's Theorem 3 to the case that \( A \) is bounded above.

The following generalization of Theorem 1 can be proved in the same way.

**Theorem 2.** \( VTLV \leq 0 \) is oscillatory if for arbitrary \( b \geq a \) there exists a number \( c \) \((c > b)\), an integer \( i \), and a piecewise \( C^1 \) function \( \phi \) on \([b, c] \) such that \( \phi(b) = \phi(c) = 0 \) and

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\[
\int_b^c \left\{ \phi'^2(x) A_{ii}(x) - \phi^2(x) B_{ii}[x, V(x), V'(x)] \right\} dx < 0
\]

for every differentiable matrix \( V(x) \) with \( \det V(x) \neq 0 \) on \([b, \infty)\).

**References**


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